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ON THE GENERALIZED SADOVSKII FUNCTOR [1]

Manuel González

Departamento de Matemáticas, Universidad de Cantabria

39005 Santander (Spain)

and

Antonio Martinón

Departamento de Análisis Matemático, Universidad de La Laguna

38071 La Laguna, Tenerife (Spain)

ABSTRACT

For an ideal of operators $\mathcal{A}$, we introduce a functor $P_{\mathcal{A}}$ which associates to a Banach space X another Banach space $P_{\mathcal{A}}(X)$ and to a (continuous linear) operator T another operator $P_{\mathcal{A}}(T)$. Particular cases of the functor $P_{\mathcal{A}}$ have been considered by several authors. By using a generalized measure of noncompactness we obtain an expression of the norm of an element of $P_{\mathcal{A}}(X)$. Moreover characterize the tauberian operators as those operators for $P_{WCo}(T)$ is one-one, WCo the ideal of weakly compact operators.

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INTRODUCTION

Independently, F. Quigley (see [RI, pp. 25-27]) and S. Berberian [BE] esentially considered for a Banach space X the quotient of the space of all bounded sequences in X, $\ell_\infty(X)$, by the subspace of all sequences converging to 0, $c_o(X)$:

$$Q(X) := \ell_\infty(X)/c_o(X) .$$

The norm of a coset $(x_n)+c_o(X)$ in Q(X) has the expresion [HA1]

$$\|(x_n)+c_o(X)\| = \limsup \|x_n\| .$$

If $T: X \longrightarrow Y$ is an operator, then T is induces another operator

$$Q(T): Q(X) \longrightarrow Q(Y) .$$

R.E. Harte [HA1] calls Q the **Berberian-Quigley functor**. For applications see [BE], [BHW], [HA1], [HA2], [RI].

B.N. Sadovskii [SA], unawer of the works of F. Quigley and S. Berberian, defines the functor P taking the subspace of $\ell_\infty(X)$ of all sequences with relatively compact range m(X). Then

$$P(X) := \ell_\infty(X)/m(X) .$$

The norm of a coset $(x_n)+m(X)$ in P(X) has the expression [HW]

$$\|(x_n)+m(X)\| = h(\{x_n\}) ,$$

where $h(\{x_n\})$ is the Hausdorff measure of noncompactness [BG] of $\{x_n\}$, the range of (x_n). If $T: X \longrightarrow Y$ is an operator, then T induces another operator

$$P(T): P(X) \longrightarrow P(Y) .$$

We call P the **Sadovskii functor**. Independently, it has been considered by J.J. Buoni, R. Harte and T. Wickstead [BHW]. For applications see [AKPS], [BHW], [CW], [FA], [HW], [SA], [TY], [ZE].

J.J. Buoni and A. Klein [BK] define the functor P^W taking the subspace of $\ell_\infty(X)$ of all sequences of relatively weakly compact range, $m^W(X)$. Then

$$P^W(X) := \ell_\infty(X)/m^W(X) .$$

Analogously to Q and P, if $T: X \longrightarrow Y$ is an operator, then we can define another operator

$$P^W(T): P^W(X) \longrightarrow P^W(Y) .$$

In this paper we show that the norm of a coset $(x_n)+m^w(X)$ in $P^w(X)$ has the expression

$$\|(x_n)+m^w(X)\| = w(\{x_n\}) ,$$

where w is the measure weak of noncompactness, defined by F.S. De Blasi [DB]. We obtain this result as a particular case of a general construction in which, for every operator ideal $\mathcal{A}$, we introduce a **generalized Sadovskii functor**. We study this functor by means of a set measure introduced by K. Astala. We also obtain a characterization of tauberian operators by means of the functor associated to the weakly compact operators.

THE ASTALA MEASURE

Recall that a class $\mathcal{A}$ of (linear and continuous) operators between Banach spaces (over the real or the complex field $\mathbb{K}$) is an **ideal of operators** if the following conditions hold [PI]:

(1) The identity operator of $\mathbb{K}$, $I_{\mathbb{K}}$, belongs to $\mathcal{A}$.

(2) The class $\mathcal{A}(X,Y)$ of all operators in $\mathcal{A}$ between the Banach spaces X and Y is a subspace of $\mathcal{L}(X,Y)$, the class of all operators between X and Y.

(3) If there exists RST, where R and T are operators, and S is an operator in $\mathcal{A}$, then RST belongs to $\mathcal{A}$.

An ideal $\mathcal{A}$ is called **closed** if each component $\mathcal{A}(X,Y)$ is closed in $\mathcal{L}(X,Y)$. $\mathcal{A}$ is called **surjective** if for any surjective operator $Q\in\mathcal{L}(Z,X)$, an operator $T\in\mathcal{L}(X,Y)$ belongs to $\mathcal{A}$ whenever $TQ\in\mathcal{A}(Z,Y)$. We use the notation $\mathcal{A}^\wedge$ for the smallest surjective closed ideal containing $\mathcal{A}$.

In the following $\mathcal{A}$ will be an ideal of operators; X, Y, Z Banach spaces; $\mathbb{B}$ the class of all Banach spaces; B_X the closed unit ball of $X\in\mathbb{B}$.

By using an ideal of operators $\mathcal{A}$, K. Astala has given the following definition:

1 DEFINITION [AS; 3.1]

Let $D \subset X$ bounded. Then $h_{\mathcal{A}}(D)$ is defined by

$$h_{\mathcal{A}}(D) := \inf \{\varepsilon>0 : \exists Z\in\mathbb{B}, \exists K\in\mathcal{A}(Z,X), D\subset KB_Z+\varepsilon B_X\} .$$

The set D is called $\mathcal{A}$-compact if $h_{\mathcal{A}}(D)=0$.

We list in the following proposition the properties of $h_{\mathcal{A}}$ that we use in this paper.

2 PROPOSITION [AS; 3.3 (b), (c); 3.5; 3.11]

Assume C, D are bounded subsets of X and $\lambda\in\mathbb{K}$ is a scalar. Then:

(1) $h_{\mathcal{A}}(C+D) \le h_{\mathcal{A}}(C)+h_{\mathcal{A}}(D)$

(2) $h_{\mathcal{A}}(\lambda D) = |\lambda|\ h_{\mathcal{A}}(D)$

(3) $h_{\mathcal{A}} = h_{\mathcal{A}^\wedge}$

(4) $h_{\mathcal{A}}(D)=0 \iff \exists Z\in\mathbb{B}, \exists K\in\mathcal{A}^\wedge(Z,X), D\subset KB_Z$

THE GENERALIZED SADOVSKII FUNCTOR

For a Banach space X we consider the space of all bounded sequences (x_n) in X:

$$\ell_\infty(X) := \{(x_n)\subset X : (x_n) \text{ bounded}\} ,$$

attached with the norm

$$\|(x_n)\| := \sup \{\|x_n\| : n\in\mathbb{N}\} .$$

We consider the subclass of the sequences (x_n) with range $\{x_n\} := \{x_n : n\in\mathbb{N}\}$ $\mathcal{A}$-compact:

$$m_{\mathcal{A}}(X) := \{(x_n)\in\ell_\infty(X) : h_{\mathcal{A}}(\{x_n\})=0\} .$$

3 PROPOSITION

$m_{\mathcal{A}}(X)$ is a closed subspace of $\ell_\infty(X)$.

PROOF. By using Proposition 2 (1), (2), we obtain that $m_{\mathcal{A}}(X)$ is a subspace of $\ell_\infty(X)$.

Now we show that $m_{\mathcal{A}}(X)$ is closed. From the definition it follows clearly that $h_{\mathcal{A}}((x_n))\leq\|(x_n)\|$. Then $h_{\mathcal{A}}: \ell_\infty(X) \longrightarrow \mathbb{R}$ is continuous, where $\mathbb{R}$ is the real field; hence $m_{\mathcal{A}}(X)=h_{\mathcal{A}}^{-1}(0)$ is closed. ■

The following proposition assures that any operator $T: X \longrightarrow Y$ maps sequences of $m_{\mathcal{A}}(X)$ into $m_{\mathcal{A}}(Y)$.

4 PROPOSITION

If $T\in\mathcal{L}(X,Y)$, then

$$(x_n)\in m_{\mathcal{A}}(X) \Rightarrow (Tx_n)\in m_{\mathcal{A}}(Y) .$$

PROOF. As $(x_n)\in m_{\mathcal{A}}(X)$, there exist $Z\in\mathbb{B}$ and $K\in\mathcal{A}^{\wedge}(Z,X)$ such that

$$\{x_n\} \subset KB_Z$$

(Proposition 2 (4)), and consequently $\{Tx_n\}\subset TKB_Z$. Since $TK\in\mathcal{A}^{\wedge}(Z,Y)$, we conclude $(Tx_n)\in m_{\mathcal{A}}(Y)$. ■

Now we can define a generalized Sadovskii functor, which also generalize the Buoni-Klein functor (see Introduction).

5 DEFINITION

We define the **generalized Sadovskii functor** $P_{\mathcal{A}}$ associated to an ideal of operators $\mathcal{A}$ in the following way

(1) $X\in\mathbb{B} \longrightarrow P_{\mathcal{A}}(X) := \ell_\infty(X)/m_{\mathcal{A}}(X)$,

(2) $T\in\mathcal{L}(X,Y) \longrightarrow P_{\mathcal{A}}(T)((x_n)+m_{\mathcal{A}}(X)) := (Tx_n)+m_{\mathcal{A}}(Y)$,

where

$$P_{\mathcal{A}}(T)((x_n)+m_{\mathcal{A}}(X)) := (Tx_n)+m_{\mathcal{A}}(Y) .$$

By Proposition 3, $P_{\mathcal{A}}(X)$ is a Banach space. Moreover,

$$T\in\mathcal{L}(X,Y) \Rightarrow P_{\mathcal{A}}(T)\in\mathcal{L}(P_{\mathcal{A}}(X),P_{\mathcal{A}}(Y)) .$$

In fact, let $(x_n)\in\ell_\infty(X)$ be such that $\|(x_n)\|\leq1$. Then

$$\|P_{\mathcal{A}}(T)((x_n)+m_{\mathcal{A}}(X))\| = \|(Tx_n)+m_{\mathcal{A}}(Y)\| \leq \|(Tx_n)\| =$$

$= \sup \{\|Tx_n\| : n\in\mathbb{N}\} \leq \|T\|$.

Hence $P_{\mathcal{A}}(T)\in\mathcal{L}(P_{\mathcal{A}}(X),P_{\mathcal{A}}(Y))$ and $\|P_{\mathcal{A}}(T)\| \leq \|T\|$.

By using the definition of $P_{\mathcal{A}}$ we obtain the following proposition.

6 PROPOSITION

Assume $T,S\in\mathcal{L}(X,Y)$, $R\in\mathcal{L}(Y,Z)$ and $\lambda\in\mathbb{K}$. Then

(1) $P_{\mathcal{A}}(T+S) = P_{\mathcal{A}}(T)+P_{\mathcal{A}}(S)$

(2) $P_{\mathcal{A}}(\lambda T) = \lambda\ P_{\mathcal{A}}(T)$

(3) $P_{\mathcal{A}}(RT) = P_{\mathcal{A}}(R)P_{\mathcal{A}}(T)$

Now we give an expression for the norm of a coset $(x_n)+m_{\mathcal{A}}(X)$ in $P_{\mathcal{A}}(X)$ by using the Astala measure $h_{\mathcal{A}}$.

7 PROPOSITION

If $(x_n)\in\ell_\infty(X)$, then

$$h_{\mathcal{A}}(\{x_n\}) = \|(x_n)+m_{\mathcal{A}}(X)\|$$

PROOF. If $\varepsilon>h_{\mathcal{A}}(\{x_n\})$, then there exist $Z\in\mathbb{B}$ and $K\in\mathcal{A}(Z,X)$ such that

$$\{x_n\} \subset KB_Z+\varepsilon B_X ,$$

hence for every $n\in\mathbb{N}$ there are $z_n\in B_Z$ and $b_n\in B_X$ verifying

$$x_n = Kz_n+\varepsilon b_n ,$$

and consequently $\|x_n-Kz_n\|\leq\varepsilon$, being $(Kz_n)\in m_{\mathcal{A}}(X)$. Then $\|(x_n)+m_{\mathcal{A}}(X)\|\leq\varepsilon$, hence

$$\|(x_n)+m_{\mathcal{A}}(X)\| \leq h_{\mathcal{A}}(\{x_n\}) .$$

If $\varepsilon>\|(x_n)+ m_{\mathcal{A}}(X)\|$, there exists $(y_n)\in m_{\mathcal{A}}(X)$ such that

$$\|(x_n-y_n)\| < \varepsilon ,$$

hence for every $n\in\mathbb{N}$, $x_n-y_n\in\varepsilon B_X$, and consequently

$$\{x_n\} \subset \{y_n\}+\varepsilon B_X \subset KB_Z+\varepsilon B_X ,$$

being $Z\in\mathbb{B}$, $K\in\mathcal{A}^{\wedge}(Z,X)$, $\{y_n\}\subset KB_Z$ (Proposition 2 (4)). Then $h_{\mathcal{A}}(\{x_n\})\leq\varepsilon$, hence

$$h_{\mathcal{A}}(\{x_n\})\leq\|(x_n)+ m_{\mathcal{A}}(X)\| .$$ ■

8 REMARK

If $\mathcal{A}$=Co, the compact operators, we obtain that h_{Co} is the Hausdorff measure of noncompactness [BG]; m_{Co} is the class of all bounded sequences for which every subsequence has a convergent subsequence; P_{Co} is the Sadovskii functor [SA].

$P_{Co}(T)$ is one-one if and only if T is upper semi-Fredholm operator (finitedimensional kernel and closed range). Moreover, $P_{Co}(T)=0$ if and only if $T \in Co$. The Proposition 7 for $\mathcal{A}$=Co appears in [HW].

9 REMARK

If $\mathcal{A}$=WCo, the weakly compact operators, we obtain what h_{WCo} is the measure of weak noncompactness, defined by De Blasi [DB]; m_{WCo} is the class of all bounded sequences for which every subsequence has a weakly convergent subsequence. The expression

$$h_{WCo}(\{x_n\}) = \|(x_n)+m_{WCo}(X)\|$$

has not appeared in the literature.

P_{WCo} is the **Buoni-Klein functor** [BK]. Our proof of Proposition 3 is more simple than the proof of [BK] for m_{WCo}.

Moreover [BK; Theorem 6] affirms the following:

Kernel of T reflexive and complemented, and range of T closed ⇒

⇒ $P_{WCo}(T)$ one-one ⇒

⇒ T have reflexive kernel

This result is weaker than Proposition 11 below. Also $P_{WCo}(T)=0$ if and only if $T \in WCo$.

CHARACTERIZATION OF TAUBERIAN OPERATORS

Let $T^* \in \mathcal{L}(Y^*,X^*)$ be the conjugate operator of $T \in \mathcal{L}(X,Y)$ and J(X) the canonical image of X in the second dual X^{**}. An operator $T \in \mathcal{L}(X,Y)$ is said to be **tauberian** if its second conjugate T^{**} verifies

$$T^{**-1}J(Y) = J(X) .$$

Tauberian operators were introduced by N. Kalton and A. Wilansky [KW]. We use the following characterization of tauberian operators [GO]: $T\in\mathcal{L}(X,Y)$ is tauberian if and only if given a bounded sequence (x_n), (Tx_n) weakly convergent implies (x_n) has a weakly convergent subsequence. We give a new characterization of tauberian operators by using the functor P_{WCo}.

12 PROPOSITION

If $T\in\mathcal{L}(X,Y)$, then

$$T \text{ tauberian} \Leftrightarrow P_{WCo}(T) \text{ one-one}$$

PROOF. If T is tauberian and

$$P_{WCo}(T)((x_n)+m_{WCo}(X)) = m_{WCo}(Y) ,$$

then $T(x_n)\in m_{WCo}(Y)$. Therefore, any subsequence of (x_n) has a subsequence weakly convergent. Hence $(x_n)\in m_{WCo}(X)$. Conversely, if (Tx_n) is weakly convergent, then $T(x_n)\in m_{WCo}(Y)$ and consequently $(x_n)\in m_{WCo}(X)$. Therefore there exists a subsequence of (x_n) which is weakly convergent and then T is tauberian. ■

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