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# A Szegö quadrature formula for a trigonometric polynomial modification of the Lebesgue measure *

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**Abstract**

Szegö quadrature formulas are used for the computation of integrals over the unit circle. They share some properties with the classical Gauss quadrature formulas for integrals on the real line. Indeed, Szegö quadrature formulas have maximum domain of validity. Furthermore, as Gauss quadrature formulas, they have positive coefficients, and nodes located in the region of integration. Nevertheless, unlike classical Gauss quadrature formulas, Szegö quadrature formulas are para-orthogonal rather than orthogonal.

There are only a few known examples of Szegö quadrature formulas. In this note a new Szegö quadrature formula for a trigonometric polynomial modification of the Lebesgue measure on the unit circle is constructed.

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# 1 Introduction

We write $\mathcal{T} = \{z \in \mathbb{C} : |z| = 1\}$ for the unit circle.

Jones, Njåstad and Thron studied in [8] the so-called Szegö quadrature formulas for the computation of integrals over the unit circle $\mathcal{T}$, that is, integrals of the form

$$(1) \quad I(f) = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) d\psi(\theta)$$

where $\psi$ is a distribution function (real valued, bounded and non-decreasing ) on $(-\pi, \pi)$. The construction of Szegö formulas is described below.

Let $(p, q)$ be a pair of integers where $p \leq q$. We denote by $\Lambda_{p,q}$ the linear space of all functions of the form $\sum_{j=p}^{q} c_j z^j$, $c_j \in \mathbb{C}$. The functions of $\Lambda_{p,q}$ are called Laurent polynomials or briefly L-polynomials. We write $\Lambda$ for the linear space of all L-polynomials. Consider the inner product on $\Lambda \times \Lambda$ given by

$$(2) \quad (f, g) = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) \overline{g(e^{i\theta})} d\psi(\theta).$$

Let $\{\varrho_n\}_0^\infty$ be the sequence of polynomials obtained by orthogonalization of $\{z^n\}_0^\infty$ with respect to the inner product (2). The sequence $\{\varrho_n\}_0^\infty$ is the sequence of Szegö polynomials with respect to the distribution function $\psi$. As it is well known, see, e.g., [9], $\varrho_n$ has its zeros in the region $|z| < 1$. Thus they are not adequate as nodes for a general purpose quadrature formula to approximate integrals over the unit circle. Quadrature formulas with nodes not in $\mathcal{T}$ are of interest for functions with poles near but not in $\mathcal{T}$. Taking the poles as nodes is the underlying idea in the method of subtract out singularities [13].

**Theorem 1** *[8] Let $\{\varrho_n\}_0^\infty$ be the sequence of Szegö polynomials with respect to the distribution function $\psi$. Let $\{\kappa_n\}_0^\infty$ be a sequence of complex numbers satisfying $|\kappa_n| = 1,\ n \geq 0$. Let $B_n(z, \kappa_n) = \varrho_n(z) + \kappa_n \varrho_n^*(z)$ where $\varrho_n^*(z) = z^n \overline{\varrho}_n(1/z)$. Then $B_n(z, \kappa_n)$ has $n$ distinct zeros $\zeta_m^{(n)}(\kappa_n)$ located on $\mathcal{T}$. Let*

$$\lambda_m^{(n)}(\kappa_n) = \int_{-\pi}^{\pi} \frac{B_n(z, \kappa_n)}{\left(z - \zeta_m^{(n)}(\kappa_n)\right) B_n'\left(\zeta_m^{(n)}(\kappa_n), \kappa_n\right)} d\psi(\theta),\ 1 \leq m \leq n.$$

*Then*

$$(3) \quad I(f) = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) d\psi(\theta) = \sum_{m=1}^{n} \lambda_m^{(n)}(\kappa_n) f\left(\zeta_m^{(n)}(\kappa_n)\right)$$

*for all $f \in \Lambda_{-(n-1),n-1}$. It holds $\lambda_m^{(n)}(\kappa_n) > 0,\ 1 \leq m \leq n,\ n \geq 1$, and the quadrature formula (3) gives the largest domain of validity, that is, there cannot exist an n-point quadrature formula $\mu(f) = \sum_{m=1}^{n} \lambda_m f(\alpha_m)$, $\alpha_m \in \mathcal{T}$ which correctly integrates any function $f \in \Lambda_{-(n-1),n}$ or any function $f \in \Lambda_{-n,n-1}$.*

The polynomials $B_n(z, \kappa_n)$, $n \geq 0$ are the para-orthogonal polynomials with respect to the distribution function $\psi$.

Thus Szegö quadrature formulas share some properties with the classical Gauss quadrature formulas for integrals on the real line. Indeed, Szegö quadrature formulas have maximum domain of validity, now in the sapce of the Laurent polynomials. Furthermore, as Gauss quadrature formulas, they have positive coefficients, and nodes located in the region of integration. Nevertheless, unlike classical Gauss quadrature formulas, Szegö quadrature formulas are para-orthogonal rather than orthogonal. One should take into account that Gauss quadrature formulas (maximum domain of exactness) for certain rational spaces of functions are not orthogonal [5] with respect to a fixed distribution function.

Due to the difficulties in the construction of Szegö quadrature formulas, interpolatory quadrature formulas on the unit circle arise as alternative. They were introduced in [2] for integrals on the unit circle. The interpolatory quadrature formulas with uniformly distributed nodes on the unit circle become the most popular. Numerical experiments and results [7, 11, 12] show that these interpolatory quadrature formulas are competitive with Szegö formulas. In addition, the nodes for this quadrature formula are easily computable, uniformly distributed on $\mathcal{T}$, and the coefficients can be efficiently computable by means of the *Fast Fourier Transform* algorithm, [11]. This facts make this interpolatory quadrature formulas suitable for practical computations.

At the beginning [8], Szegö quadrature formulas were constructed as a tool for the solution of the trigonometric moment problem. In [13], both interpolatory and Szegö quadrature formulas were used as part of efficient quadrature formulas for the computation of integrals with Poisson type kernel that appear in the solution of boundary value problems for a circle.

Szegö quadrature formulas have been included in the more general topic of rational Szegö quadrature formulas [1].

There are only a few known examples of Szegö quadrature formulas. Among them, for the Lebesgue measure [3], for the Poisson integral [14], for rational modifications of the Lebesgue measure on the unit circle [7], for a certain measure connected with $q-$starlike functions [10], and for Jacobi type weight function on the unit circle [4]. Next we construct a one parametric family of Szegö quadrature formulas for the trigonometric polynomial modification of the Lebesgue measure on the unit circle given by

$$d\psi(\theta) = \left|e^{i\theta} - \beta\right|^2 d\theta, \; \beta \in \mathbb{C}, \; -\pi \leq \theta < \pi.$$

The corresponding orthogonal polynomials were constructed in [6]. The associated moments

$$m_k = I\left(z^k\right) = \int_{-\pi}^{\pi} e^{ik\theta} \left|e^{i\theta} - \beta\right|^2 d\theta, \; k \in \mathbb{Z}$$

are given by

$$(4) \quad m_0 = 2\pi\left(1 + |\beta|^2\right), \; m_1 = -2\pi\beta, \; m_{-1} = -2\pi\overline{\beta}, \; m_k = 0, \; |k| \geq 2, \; k \in \mathbb{Z}.$$

## 2 Construction of the quadrature formula

First we deal with the case that $\beta$ lies on the unit circle, that is, $\beta \in \mathbb{C}$, $|\beta| = 1$. Without loss of generality, and for simplicity, we will take $\beta = 1$. Indeed, if $\beta \in \mathbb{C}$, $|\beta| = 1$ and $\beta \neq 1$ then we can make an angle rotation on the complex unit circle.

The corresponding orthogonal polynomials $\varrho_n(z)$ are given by [6],

$$\varrho_n(z) = \sum_{k=0}^{n} \frac{k+1}{n+1} z^k, \; n \geq 0.$$

Hence the para-orthogonal polynomials $B_n(z, \kappa_n) = \varrho_n(z) + \kappa_n \varrho_n^*(z)$, $n \geq 1$ where as usual, $\kappa_n \in \mathbb{C}$, $|\kappa_n| = 1$, and $\varrho_n^*(z) = z^n \overline{\varrho}_n(1/z)$ are given, for fixed $\kappa_n = 1$ by

$$(5) \quad B_n(z, 1) = \frac{n+2}{n+1} \sum_{k=0}^{n} z^k = \frac{n+2}{n+1} \frac{1 - z^{n+1}}{1 - z}.$$

The nodes $\zeta_m^{(n)}(1)$, $1 \leq m \leq n$, $n \geq 1$ of the $n$ point Szegö quadrature formula ($\kappa_n = 1$, $n \geq 1$) are the $n$ roots of $B_n(z, 1)$, and its coefficients $\lambda_m^{(n)}(1)$, $1 \leq m \leq n$, $n \geq 1$ are given by

$$\lambda_m^{(n)}(1) = \int_{-\pi}^{\pi} L_m^{(n)}\left(e^{i\theta}\right) \left|e^{i\theta} - 1\right|^2 d\theta$$

where

$$L_m^{(n)}(z) = \frac{B_n(z, 1)}{\left(z - \zeta_m^{(n)}(1)\right) B_n'\left(\zeta_m^{(n)}(1), 1\right)} \in \Lambda_{0,n-1}.$$

From (4), and since $\beta = 1$, we get that $m_0 = 4\pi$, $m_{-1} = m_1 = -2\pi$ and $m_k = 0$, $|k| \geq 2$. Thus

$$\begin{aligned} \lambda_m^{(n)}(1) &= L_m^{(n)}(0) m_0 + \left(L_m^{(n)}\right)'(0) m_1 \\ &= 2\pi \left( -\frac{2B_n(0,1)}{\zeta_m^{(n)}(1) B_n'\left(\zeta_m^{(n)}(1), 1\right)} + \frac{B_n'(0,1)\zeta_m^{(n)}(1) + B_n(0,1)}{\left(\zeta_m^{(n)}(1)\right)^2 B_n'\left(\zeta_m^{(n)}(1), 1\right)} \right). \end{aligned}$$

Since

$$B_n(0,1) = B_n'(0,1) = \frac{n+2}{n+1},$$

it holds

$$\lambda_m^{(n)}(1) = \frac{2\pi(n+2)\left(1 - \zeta_m^{(n)}(1)\right)}{(n+1)\left(\zeta_m^{(n)}(1)\right)^2 B_n'\left(\zeta_m^{(n)}(1), 1\right)}.$$

From (5) and taking into account that $1-\left(\zeta_m^{(n)}(1)\right)^{n+1}=0$ we get

$$B_n'\left(\zeta_m^{(n)}(1),1\right)=-\frac{(n+2)\left(\zeta_m^{(n)}(1)\right)^n}{1-\zeta_m^{(n)}(1)},$$

and hence

$$(6)\quad \lambda_m^{(n)}(1)=-\frac{2\pi\left(1-\zeta_m^{(n)}(1)\right)^2}{(n+1)\left(\zeta_m^{(n)}(1)\right)^{n+2}}=-\frac{2\pi\left(1-\zeta_m^{(n)}(1)\right)^2}{(n+1)\zeta_m^{(n)}(1)}.$$

One has that $B_n\left(\zeta_m^{(n)}(1),1\right)=0$, or equivalently

$$1+\zeta_m^{(n)}(1)+\left(\zeta_m^{(n)}(1)\right)^2+\cdots+\left(\zeta_m^{(n)}(1)\right)^n=0.$$

Since $\zeta_m^{(n)}(1)\neq 0$, and $\zeta_m^{(n)}(1)\neq 1$ we get

$$(7)\quad \frac{1}{\zeta_m^{(n)}(1)}=-\left(1+\zeta_m^{(n)}(1)+\left(\zeta_m^{(n)}(1)\right)^2+\cdots+\left(\zeta_m^{(n)}(1)\right)^{n-1}\right)=-\frac{1-\left(\zeta_m^{(n)}(1)\right)^n}{1-\zeta_m^{(n)}(1)}.$$

From (6), (7), and taking into account that $\left(\zeta_m^{(n)}(1)\right)^{n+1}=1$ we deduce

$$\begin{aligned}\lambda_m^{(n)}(1) &= \frac{2\pi}{n+1}\left(1-\left(\zeta_m^{(n)}(1)\right)^n\right)\left(1-\zeta_m^{(n)}(1)\right)\\ &= \frac{2\pi}{n+1}\left(2-\left(\zeta_m^{(n)}(1)+\left(\zeta_m^{(n)}(1)\right)^n\right)\right)\\ &= \frac{2\pi}{n+1}\left(2-\left(\zeta_m^{(n)}(1)+\overline{\zeta_m^{(n)}(1)}\right)\right).\end{aligned}$$

For the last equality take into account that $\left(\zeta_m^{(n)}(1)\right)^n=\frac{1}{\zeta_m^{(n)}(1)}=\overline{\zeta_m^{(n)}(1)}$ since $\left|\zeta_m^{(n)}(1)\right|=1$. Then

$$\lambda_m^{(n)}(1)=\frac{4\pi}{n+1}\left(1-\operatorname{Re}\left(\zeta_m^{(n)}(1)\right)\right).$$

Thus we have obtained that the $n$ point Szegö quadrature formula for the distribution function

$$d\psi(\theta)=\left|e^{i\theta}-1\right|^2 d\theta,\ -\pi\leq\theta<\pi,$$

and $\kappa_n = 1$ has nodes $\zeta_m^{(n)}(1)$ and coefficients $\lambda_m^{(n)}(1)$ given by

$$\left.\begin{aligned} \zeta_m^{(n)}(1) &= e^{2\pi mi/(n+1)} \\ \lambda_m^{(n)}(1) &= \frac{4\pi}{n+1}\left(1-\cos\left(\frac{2\pi m}{n+1}\right)\right) \end{aligned}\right. \quad 1 \le m \le n,\ n \ge 1.$$

This $n$ point Szegö quadrature formula satisfies

$$\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)\left|e^{i\theta}-1\right|^2 d\theta = \sum_{m=1}^{n} \lambda_m^{(n)}(\kappa_n) f\left(\zeta_m^{(n)}(\kappa_n)\right),\ f \in \Lambda_{-(n-1),n-1}.$$

Next we consider the remain case

$$d\psi(\theta) = \left|e^{i\theta}-\beta\right|^2 d\theta,\ \beta \in \mathbb{C},\ |\beta| \neq 1.$$

Without loss of generality, and for simplicity, we will take $\beta \in \mathbb{R}$, $\beta \ge 0$, and $\beta \neq 1$. Otherwise we can make an angle rotation on the complex unit circle. The corresponding orthogonal polynomials are given by [6],

$$\varrho_n(z) = \frac{1}{\beta^{2(n+1)}-1}\sum_{k=0}^{n} \beta^k \left(\beta^{2(n-k+1)}-1\right) z^{n-k}.$$

After several elementary calculations is deduced that the para-orthogonal polynomials, for fixed $\kappa_n = 1$, $n \ge 1$ are given by

$$B_n(z,1) = \frac{\beta^{n+2}-1}{\beta^{2(n+1)}-1}\left(\frac{1-(\beta z)^{n+1}}{1-\beta z}+\frac{\beta^{n+1}-z^{n+1}}{\beta - z}\right).$$

The nodes $\zeta_m^{(n)}(1)$, $1 \le m \le n$, $n \ge 1$ of the $n$ point Szegö quadrature formula ($\kappa_n = 1$, $n \ge 1$) are the $n$ roots of $B_n(z,1)$, and its coefficients $\lambda_m^{(n)}(1)$, $1 \le m \le n$, $n \ge 1$ are given by

$$\lambda_m^{(n)}(1) = \int_{-\pi}^{\pi} L_m^{(n)}\left(e^{i\theta}\right)\left|e^{i\theta}-\beta\right|^2 d\theta$$

where

$$L_m^{(n)}(z) = \frac{B_n(z,1)}{\left(z-\zeta_m^{(n)}(1)\right) B_n'\left(\zeta_m^{(n)}(1),1\right)} \in \Lambda_{0,n-1}.$$

Thus

$$\begin{aligned}\lambda_m^{(n)}(1) &= L_m^{(n)}(0)m_0 + \left(L_m^{(n)}\right)'(0)m_1\\ &= -\frac{2\pi\left(1+\beta^2\right)B_n(0,1)}{\zeta_m^{(n)}(1)B_n'\left(\zeta_m^{(n)}(1),1\right)} + \frac{2\pi\beta\left(B_n'(0,1)\zeta_m^{(n)}(1)+B_n(0,1)\right)}{\left(\zeta_m^{(n)}(1)\right)^2 B_n'\left(\zeta_m^{(n)}(1),1\right)}\\ &= \frac{2\pi\left(\beta\zeta_m^{(n)}(1)B_n'(0,1)+B_n(0,1)\left(\beta-(1+\beta^2)\zeta_m^{(n)}(1)\right)\right)}{\left(\zeta_m^{(n)}(1)\right)^2 B_n'\left(\zeta_m^{(n)}(1),1\right)}.\end{aligned}$$

One encounter that

$$B_n(0,1)=\frac{(\beta^{n+2}-1)(1+\beta^n)}{\beta^{2(n+1)}-1},\ B_n'(0,1)=\frac{(\beta^{n+2}-1)(\beta+\beta^{n-1})}{\beta^{2(n+1)}-1},$$

and

$$B_n'\left(\zeta_m^{(n)}(1),1\right)=\frac{\beta^{n+2}-1}{\beta^{2(n+1)}-1}\frac{C_1}{C_2}$$

where

$$\begin{aligned}C_1 &= (n+2)(\beta^{n+1}+\beta)\left(\zeta_m^{(n)}(1)\right)^{n+1}\\ &\quad -(n+1)(\beta^{n+2}+1)\left(\zeta_m^{(n)}(1)\right)^n-(\beta^{n+2}+1),\\ C_2 &= \left(1-\beta\zeta_m^{(n)}(1)\right)\left(\beta-\zeta_m^{(n)}(1)\right).\end{aligned}$$

Thus

$$\lambda_m^{(n)}(1)=\frac{2\pi C_2C_3}{\left(\zeta_m^{(n)}(1)\right)^2 C_1}$$

where

$$C_3=\zeta_m^{(n)}(1)\beta\left(\beta+\beta^{n-1}\right)+\left(1+\beta^n\right)\left(\beta-\left(1+\beta^2\right)\zeta_m^{(n)}(1)\right).$$

Taking into account that $B_n\left(\zeta_m^{(n)}(1),1\right)=0$, and $\left|\zeta_m^{(n)}(1)\right|^2=\zeta_m^{(n)}(1)\overline{\zeta_m^{(n)}(1)}=1$ one can deduce that

$$\lambda_m^{(n)}(1)=\frac{2\pi\left(-1+2\beta\mathrm{Re}\left(\zeta_m^{(n)}(1)\right)-\beta^2\right)\left(-(\beta^{n+1}+\beta)+(\beta^{n+2}+1)\overline{\zeta_m^{(n)}(1)}\right)}{(n+2)(\beta^{n+1}+\beta)-(n+1)(\beta^{n+2}+1)\overline{\zeta_m^{(n)}(1)}-(\beta^{n+2}+1)\left(\overline{\zeta_m^{(n)}(1)}\right)^{n+1}}.$$

Multiply numerator and denominator by $-(\beta^{n+1}+\beta)+(\beta^{n+2}+1)\zeta_m^{(n)}(1)$. After some elementary calculations is deduced that the coefficients $\lambda_m^{(n)}(1),\ 1\le m\le n,\ n\ge 1$ are given by

$$\lambda_m^{(n)}(1)=\frac{2\pi C_4 C_5}{C_6} \tag{8}$$

where

$$\begin{aligned}
C_4 &= -1+2\beta\mathrm{Re}\left(\zeta_m^{(n)}(1)\right)-\beta^2,\\
C_5 &= \left(\beta^{n+1}+\beta\right)^2-\left(\beta^{n+1}+\beta\right)\left(\beta^{n+2}+1\right)2\mathrm{Re}\left(\zeta_m^{(n)}(1)\right)+\left(\beta^{n+2}+1\right)^2,\\
C_6 &= -(n+2)\left(\beta^{n+1}+\beta\right)^2-n\left(\beta^{n+2}+1\right)^2\\
&\quad +(n+1)\left(\beta^{n+1}+\beta\right)\left(\beta^{n+2}+1\right)2\mathrm{Re}\left(\zeta_m^{(n)}(1)\right).
\end{aligned}$$

Thus the $n$ point Szegö quadrature formula for the distribution function

$$d\psi(\theta)=\left|e^{i\theta}-\beta\right|^2 d\theta,\ \beta\in\mathbb{R},\ \ \beta\ge 0,\ \beta\ne 1,$$

and for $\kappa_n=1,\ n\ge 1$ is given as follows. Its nodes $\zeta_m^{(n)}(1),\ 1\le m\le n,\ n\ge 1$ are the roots of the polynomial

$$\left(\beta^{n+1}+\beta\right)z^n+\left(\beta^n+\beta^2\right)z^{n-1}+\cdots+\left(\beta^2+\beta^n\right)z+\left(\beta+\beta^{n+1}\right)=0,$$

and the coefficients $\lambda_m^{(n)}(1),\ 1\le m\le n,\ n\ge 1$ are given by (8). This quadrature formula satisfies

$$\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)\left|e^{i\theta}-\beta\right|^2 d\theta=\sum_{m=1}^{n}\lambda_m^{(n)}(\kappa_n)f\left(\zeta_m^{(n)}(\kappa_n)\right),\ f\in\Lambda_{-(n-1),n-1}.$$

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