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ON SOME GENERATING FUNCTIONS OF HYPERGEOMETRIC POLYNOMIAL - II

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Abstract

In this paper the author establishes two theorems on bilateral and mixed trilateral generating functions of modified hypergeometric polynomial from the Lie group view point. An application of the theorem on biateral generating relation of the said polynomial is also pointed out.

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Key words and Phrases : *Generating functions, Hypergeometric polynomial; group-theoretic method.*

1. INTRODUCTION

The hypergeometric polynomial is defined as follows :

$$ {}_2F_1(-n,\alpha;\nu;x) = \sum_{k=0}^{n} \frac{(-n)_k(\alpha)_k x^k}{(\nu)_k k!} \tag{1.1}$$

Recently many researchers have derived a large number of generating functions involving Hypergeometric polynomial by using L.Weisner's [1] group theoretic method. The object of the present paper is to prove the following theorems in connection with the bilateral and mixed trilateral generating relations by giving suitable interpretation of the parameter (α) of the polynomial under consideration from Lie-group view point.

Theorem 1.

If there exists a generating relation of the form

$$ G(x,w) = \sum_{\alpha=0}^{\infty} a_\alpha\, {}_2F_1(-n,\alpha;\nu+\alpha;x)w^\alpha \tag{1.2}$$

then

$$ G\left(\frac{x+t}{1+t}, \frac{zt}{1+t}\right)$$

$$ = \sum_{\alpha=0}^{\infty} t^\alpha \sigma_\alpha(z,x) \tag{1.3}$$

where

$$\sigma_\alpha(z,x)=\sum_{k=0}^{\alpha}(-1)^{\alpha-k}\frac{a_k(k)_{\alpha-k}^{(\nu+n+k)}\alpha-k}{(\alpha-k)!(\nu+k)_{\alpha-k}}z^k\,{}_2F_1(-n,\alpha;\nu+\alpha;x)$$

Theorem-2

If there exists a generating relation of the form

$$G(x,u,w)=\sum_{\alpha=0}^{\infty}a_\alpha\,{}_2F_1(-n,\alpha;\nu+\alpha;x)g_\alpha(u)w^\alpha \tag{1.4}$$

then

$$G\left(\frac{x+t}{1+t},u,\frac{zt}{1+t}\right)=\sum_{\alpha=0}^{\infty}t^\alpha\sigma_\alpha(z,u,x) \tag{1.5}$$

where

$$\sigma_\alpha(z,u,x)\alpha=\sum_{k=0}^{\alpha}(-1)^{\alpha-k}\frac{a_k(k)_{\alpha-k}(\nu+n+k)_{\alpha-k}z^k}{(\alpha-k)!(\nu+k)_{\alpha-k}}g_\alpha(u)_2F_1(-n,\alpha;\nu+\alpha;x)$$

The importance of our above said theorems lies in the fact that one can get a large number of bilateral and mixed trilateral generating relations from (1.3) and (1.5) by attributing different suitable values to a_α in (1.2) and (1.4) respectively.

2. PROOF OF THEOREM -1

Let us define the partial differential operator R as follows :

$$R=(1-x)y\frac{\partial}{\delta x}-y^2\frac{\partial}{\delta y} \tag{2.1}$$

such that

$$R\left[{}_2F_1(-n,\alpha;\nu+\alpha;x)y^\alpha\right]=\frac{(-\alpha)(\nu+\alpha+n)}{(\nu+\alpha)}{}_2F_1(-n,\alpha+1;\nu+\alpha+1;x)y^{\alpha+1} \tag{2.2}$$

The extended form of the group generated by R is

$$\exp(wR)f(x,y)=f\left(\frac{x+wy}{1+wy},\frac{y}{1+wy}\right) \tag{2.3}$$

Let us assume the generating relation

$$G(x,w)=\sum_{\alpha=0}^{\infty}a_\alpha\,{}_2F_1(-n,\alpha;\nu+\alpha;x)w^\alpha \tag{2.4}$$

Replacing w by wyz in (2.4), we get

$$G(x,wyz)=\sum_{\alpha=0}^{\infty}a_\alpha\,{}_2F_1(-n,\alpha;\nu+\alpha;x)(wz)^\alpha y^\alpha \tag{2.5}$$

Operating both sides of (2.5) by exp (wR), we get

$\exp(wR)[G(x, wyz)]$

$$= \exp(wR)\left[\sum_{\alpha=0}^{\infty} a_{\alpha}(wz)^{\alpha}\, {}_2F_1(-n, \alpha; \nu+\alpha; x)y^{\alpha}\right] \tag{2.6}$$

Now from the left member of (2.6), we get

$$\exp(wR)[G(x, wyz)] = G\left(\frac{x+wy}{1+wy}, \frac{wyz}{1+wy}\right) \tag{2.7}$$

From the right member of (2.6) we get

$$\exp(\omega R)\left[\sum_{\alpha=0}^{\infty} a_{\alpha}\, 2F_1(-n, \alpha; \nu+\alpha; x)(\omega z)^{\alpha} y^{\alpha}\right]$$

$$= \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\infty} a_{\alpha} \frac{w^{k+\alpha} z^{\alpha}}{k!} R^{k}\left({}_2F_1(-n, \alpha; \nu+\alpha; x)y^{\alpha}\right)$$

$$= \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\infty} a_{\alpha} \frac{\omega^{k+\alpha} z^{\alpha}}{k!} \frac{(-\alpha)_k (\nu+k+n)_k}{(\nu+k)_k} \times 2F_1(-n, \alpha+k; \nu+\alpha+k; x)y^{\alpha+k}$$

$$= \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\alpha} (-1)^k . a_{\alpha-k}{}^{(\alpha-k)_k} . \frac{(wy)^{\alpha}}{k!} . \frac{(\nu+n+\alpha-k)_k}{(\nu+\alpha-k)_k} {}_2F_1(-n, \alpha; \nu+\alpha; x) z^{\alpha-k} \tag{2.8}$$

Equating (2.7) and (2.8) and then writing t for wy, we get

$$G\left(\frac{x+t}{1+t}, \frac{zt}{1+t}\right)$$

$$= \sum_{\alpha=0}^{\infty} t^{\alpha} \sigma_{\alpha}(z, x) \tag{2.9}$$

where

$$\sigma_{\alpha}(z, x) = \sum_{k=0}^{\alpha} (-1)^{\alpha-k} a_k \frac{(k)_{\alpha-k} (\nu+n+k)_{\alpha-k}}{(\alpha-k)!(\nu+k)_{\alpha-k}} {}_2F_1(-n, \alpha; \nu+\alpha; x) z^{k}$$

This completes the proof of the theorem-1.
Applying the same technique easily we can prove theorem-2.

3. APPLICATION

As an application, we consider the following generating relation [2]

$${}_2F_1\left(-n, -, \nu; \frac{x-t}{1-t}\right)$$

$$= \sum_{\alpha=0}^{\infty} \frac{(\nu+n)_\alpha}{\alpha!(\nu)_\alpha} {}_2F_1(-n,\alpha;\nu+\alpha;x)t^\alpha. \qquad (3.1)$$

If we take $a_\alpha = \dfrac{(\nu+n)_\alpha}{\alpha!(\nu)_\alpha}$, then we get

$$G(x,t) = {}_2F_1\left(-n,-;\nu;\frac{x-t}{1-t}\right). \qquad (3.2)$$

By applying our theorem-1, we get the following result :

$${}_2F_1\left(-n,-;\nu;\frac{x+(1-z)t}{1+(1-z)t}\right)$$

$$= \sum_{\alpha=0}^{\infty} t^\alpha \sigma_\alpha(z,x) \qquad (3.3)$$

where

$$\sigma_\alpha(z,x) = \sum_{k=0}^{\alpha} \frac{(-1)^{\alpha-k}}{(\alpha-k)!} \cdot \frac{(\nu+n)_k}{k!(\nu)_k} \cdot \frac{(k)_{\alpha-k}(\nu+n+k)_{\alpha-k}}{(\nu+k)_{\alpha-k}} z^k \, {}_2F_1(-n,\alpha;\nu+\alpha;x)$$

REFERENCES

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[2] Mukherjee, M.C. : Weisner's methodic study of hypergeometric polynomials, - Communicated.