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GENERALIZED SEMIFREDHOLM TRANSFORMATIONS

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Abstract. – The aim of this paper is to study normally solvable operators with superreflexive null space or superreflexive conull space. We use the super weakly compact operators to replace the compact operators of the classical theory in a natural way.

Resumen. – Utilizando técnicas de análisis no estándar, obtenemos en estas notas resultados de dualidad, perturbación y composición de operadores normalmente solubles cuyo núcleo o rango es superreflexivo.

Key words: superreflexive space, super weakly compact operator.

The spaces considered here are always assumed to be Banach spaces. For a space X, $\hat{X}$ denotes the nonstandard hull of X considered in [6]. Recall that X is superreflexive ([7]) if any space finitely representable in X is reflexive, or, equivalently, if $\hat{X}$ is reflexive ([3; th. 8.5]).

SR, R and F will denote the classes of all superreflexive, reflexive and finite dimensional Banach spaces respectively.

All operators acting between Banach spaces are supposed to be linear and bounded. The class of all operators between arbitrary Banach spaces is denoted by L. Given two Banach spaces, X and Y, L(X,Y) is the space of all operators from X into Y. For $T\in L(X,Y)$, N(T) and R(T) will denote the null space and the range of T respectively. The ascent and descent of T are respectively denoted by a(T) and d(T).

If $T\in L(X,Y)$, then the operators $\hat{T}\in L(\hat{X},\hat{Y})$ and $\bar{T}\in L(X''/J_XX, Y''/J_YY)$ are defined by $\hat{T}\hat{x}=(Tx)^\wedge$, $x\in X$, and $\bar{T}(x''+J_XX)=T''x''+J_YY$, $x''\in X''$, where J_X is the embedding map of X into X''.

An operator $T\in L(X,Y)$ is called super weakly compact ([8]), $T\in SWCo(X,Y)$, if for every positive real number r there is a positive integer n such that there do not exist finite sequences $\{x_1,\dots,x_n\}$ in X and $\{f_1,\dots,f_n\}$ in Y' with $\|x_i\|=\|f_i\|=1$ $(i=1,\dots,n)$, $f_j(Tx_i)>r$ $(1\le j\le i\le n)$, $f_j(Tx_i)=0$ $(1\le i<j\le n)$. This condition is equivalent to $\hat{T}$ weakly compact, $\hat{T}\in WCo(\hat{X},\hat{Y})$, by a result of Tacon ([8, th. 1]).

We shall consider the following operator classes:

$\Phi_{A+} = \{T\in L\colon R(T) \text{ closed}, N(T)\in A\}$

$\Phi_{A-} = \{T\in L\colon R(T) \text{ closed}, Y/R(T)\in A\}$

$\Phi_A = \Phi_{A+}\cap\Phi_{A-}$,

where A=SR or R; if A=F, we write Φ_+, Φ_- and Φ respectively.

The terminology and notation from nonstandard analysis that we use is like those of [6].

In this note we study solvable operators with superreflexive null space or superreflexive conull space. A theory similar to the classical Fredholm theory exists for the generalized Fredholm operators if we use the super weakly compact operators to replace the compact operators of the classical theory.

Now, we present two examples illustrating that the classes $\Phi_{SR+}\cup\Phi_{SR-}$ and Φ_{SR} are, in general, different.

We take from [3] the notion of weakly-p-compact space (W_p) and weakly-p-compact operator (WCo_p). The operator $T\in L(X,Y)$ is said to be weakly-p-compact ($p\geq 1$) if T sends bounded sequences into sequences which have a weakly-p-convergent subsequence. A Banach space Z belongs to W_p if I_Z belongs to $WCo_p(Z)$.

Let $\mathcal{T}$ be the Tsirelson's space. Then, if we combine the properties

(i) $\mathcal{T}$ has all the W_p properties ([3])

(ii) if Z is a superreflexive Banach space, then there are numbers $p>q>1$ such that $Z\in W_p$ but $Z\notin W_q$ ([3]),

we get that $\mathcal{T}$ is not superreflexive and that $\mathcal{T}$ does not contain a superreflexive subspace (or a superreflexive quotient).

So, if M is an infinite dimensional closed subspace of $\mathcal{T}$ and i_M, q_M denote the inclusion of M into $\mathcal{T}$ and the quotient map of $\mathcal{T}$ onto $\mathcal{T}/M$ respectively, we have that $i_M\in\Phi_{SR+}\setminus\Phi_{SR}$, $q_M\in\Phi_{SR-}\setminus\Phi_{SR}$. Since $\mathcal{T}$ is a reflexive space, it is clear that $i_M\in\Phi_{R-}\setminus\Phi_{SR-}$, $q_M\in\Phi_{R+}\setminus\Phi_{SR+}$, which shows that, in general, $\Phi_{SR+}\subsetneq\Phi_{R+}$, $\Phi_{SR-}\subsetneq\Phi_{R-}$.

Theorem 1

Let $T\in L$. Then, T belongs to Φ_{SR+}, Φ_{SR-} or Φ_{SR} if and only if T' belongs to Φ_{SR-}, Φ_{SR+} or Φ_{SR} respectively.

Proof:

It is sufficient to observe that $N(T)'\cong X'/R(T')$, $N(T')\cong(Y/R(T))'$ and that the superreflexivity is preserved under isomorphisms and duality ([7, th.2]).

Lemma 2

Let $T\in L(X,Y)$ with closed range. Then, $N(T)^{\wedge}$ is isometric to $N(\hat{T})$.

Proof:

Let π_X denote the canonical projection from fin^*X to $\hat{X}$. We consider the map $B:\pi_{N(T)}(x)\in N(T)^{\wedge}\mapsto B(\pi_{N(T)}(x))=\pi_X(x)\in N(\hat{T})$. By using the Transfer Principle, we deduce that $^*N(T)=N(^*T)$, and so, it is easy to see that B is well defined. Moreover, from the definitions it follows that B is a linear isometry. Finally, we show that B is surjective.

Let $\pi_X(x)\in N(\hat{T})$, that is, x is finite and $^*Tx\approx 0$. Since R(T) is closed, there exists M>0 such that $\|^*Tx\|\geq M\|x\|$ for $x\in {}^*X\setminus {}^*N(T)$. Thus, if $x\in {}^*N(T)$, then $\pi_X(x)=B(\pi_{N(T)}(x))$, and if $x\notin {}^*N(T)$, then $x\approx 0$; so, $\pi_X(x)=\pi_X(0)=B(\pi_{N(T)}(0))$.

Theorem 3

Let $T\in L(X,Y)$ with closed range and $K\in SWCo(X,Y)$ such that $R(T+K)$ is closed.

(i) If $N(T)\in SR$, then $N(T+K)\in SR$.

(ii) If $Y/R(T)\in SR$, then $Y/R(T+K)\in SR$.

Proof:

If R(T) is closed, then $R(\hat{T})$ is closed ([8, th. 3]). Moreover, by virtue of previous lemma, $N(T)^{\wedge}\cong N(\hat{T})$, $N(T+K)^{\wedge}\cong N((T+K)^{\wedge})$.

(i) Assume that N(T) is superreflexive. Then, $N(\hat{T})$ is reflexive, and since $\hat{K}\in WCo(\hat{X},\hat{Y})$, $N(\hat{T}+\hat{K})$ is reflexive ([10, th. 6.1 and 6.2]). Hence, N(T+K) is superreflexive.

(ii) Suppose that Y/R(T) is superreflexive. Then N(T') is superreflexive, and since $K'\in SWCo(Y',X')$ ([8, corollary 1]), from (i) we conclude that $(Y/R(T+K))'\in SR$, and consequently, Y/R(T+K) is superreflexive.

Theorem 4

Let T,S∈L.

(i) If $J\in\{\Phi_{SR+}, \Phi_{SR-}, \Phi_{SR}\}$, T,S∈J and TS is normally solvable, then TS∈J.

(ii) $\Phi_{SR+}.\Phi_{+}\subset\Phi_{SR+}$

$\Phi_{+}.\Phi_{SR+}\subset\Phi_{SR+}$

$\Phi_{-}.\Phi_{SR-}.\subset\Phi_{SR-}$

$\Phi_{SR-}\Phi_{-}\subset\Phi_{SR-}.$

Proof:

Since $T\in\{\Phi_{SR+}, \Phi_{SR-}, \Phi_{SR}\}$ if and only if $\hat{T}\in\{\Phi_{SR+}, \Phi_{SR-}, \Phi_{SR}\}$, assertion (i) follows by applying lemma 2 and some results of Yang ([10, th. 6.1, 6.3, 6.4, 6.6 and 5.3]).

To prove (ii) by virtue of (i) it is enough to verify that R(TS) is closed, which follows immediately from the observation that R(TS) is closed if and only if R(S)+N(T) is closed (for operators with closed range).

Theorem 5

Let K∈SWCo(X). Then, N(I-K) and $Y/\overline{R(I-K)}$ are superreflexive. In general, R(I-K) is not closed.

Proof:

Let i_M be the inclusion of M into X, where M is the closed subspace $K^{-1}(N(I-K))$. Then, $Ki_M\in SWCo(M,X)$ ([8, corollary 1]), $R(Ki_M)=N(I-K)$, and, hence, N(I-K)∈SR. Moreover, since K'∈SWCo(X'), we have $Y/\overline{R(I-K)}\in SR$.

Now, we define $K\in L(\ell_2)$ by $Ke_n=e_{n+1}$, $n\in\mathbb{N}$. Then, it is clear that R(I-K) is not closed and K is super weakly compact, since ℓ_2 is superreflexive ([2]).

Co(X,Y) and F(X,Y) will denote the spaces of compact and finite range operators from X into Y.

Theorem 6

Let $K\in SWCo(X)\setminus Co(X)$. Then, in general, $a(I-K)$, $d(I-K)$ are not finite.

Proof:

First, it is an easy calculation that, if $T_i\in L(X_i,Y_i)$, i=1,2, then $a(T_1xT_2)=\max\{a(T_1),a(T_2)\}$, $d(T_1xT_2)=\max\{d(T_1),d(T_2)\}$, and, since $S\in L(X,Y)$ is weakly compact if and only if $\bar{S}=0$ ([10, th. 4.1]), we deduce that T_1xT_2 is super weakly compact if and only if T_1, T_2 are super weakly compact.

Now, we show that if $K\in SWCo(X)\setminus Co(X)$, we then have the following possibilities for the pair $(a(I-K), d(I-K))$.

(1) $a(I-K)=\infty$, $d(I-K)<\infty$. Let $T\in Fi(X)$, where X is a non-superreflexive Banach space and define $S\in L(\ell_2)$ by $Se_1=e_1$, $Se_n=e_n-e_{n-1}$, $n>1$. Then, $a(I-T)=d(I-T)<\infty$, $a(I-S)=\infty$, $d(I-S)=0$. Hence, the operator $K=TxS$ satisfies (1).

(2) $a(I-K)<\infty$, $d(I-K)=\infty$. In the above case, it suffices to replace S by the conjugate operator.

(3) $a(I-K)=d(I-K)=\infty$. Let T and S be the operators as in the case (1) and $K=(TxS)x(TxS')$.

(4) $a(I-K)=d(I-K)<\infty$. Let $T\in Fi(X)$, where X is a non-superreflexive Banach space, and $S\in L(\ell_2)$ defined by $Se_1=0$, $Se_n=e_n$, $n>1$. Then, $a(I-S)=d(I-S)=1$, $a(I-T)=d(I-T)<\infty$. The operator $K=TxS$ satisfies $a(I-K)=d(I-K)<\infty$.

Remark

In [4], Davis et al. prove that every weakly compact operator factors through a reflexive space, that is, $Op(R)=WCo$, and, since $T\in SWCo$ if and only if $\hat{T}\in WCo$, it follows that $Op(SR)\subset SWCo$. However, the following example will show that the class $Op(SR)$ is, in general, properly contained in SWCo.

Consider X to be the Orlicz's space $L^{q}([0,1],dt)$, with $\phi(t)=t(1+\log(1+t))$. Since $\phi(t)\geq t$, there is a bounded injection, T, of L^{ϕ} into L^{1}. Then, it has been proved in [1] that $T\notin Op(SR)$, but $T\in SWCo$, since if T is not in SWCo, then we derive from [1, prop. I.3] that $TB_{L^{\phi}}$ has the finite tree property, which is not true ([1]).

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