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SOME INTEGRALS INVOLVING THE PRODUCT OF A CLASS OF GENERALIZED POLYNOMIALS

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ABSTRACT

In this paper we consider integrals of the product of two polynomials of the form

$$\int_0^{\infty} x^{\lambda} e^{-ax^r} G_n^{(\alpha)}(bx,\ r,\ p,\ 1)\ G_m^{(\beta)}(cx,\ r,\ q,\ 1)\ dx$$

where $\lambda > -1$, $r > 0$, $p+q < a$ and $G_n^{(\alpha)}(x,\ r,\ p,\ 1)$ is a class of unified polynomials defined by the generalized Rodrigues' formula

$$G_n^{(\alpha)}(x,\ r,\ p,\ 1) = \frac{1}{n!}\ x^{-\alpha-n}\ e^{px^r}\ \ \theta^n\left(x^{\alpha}\ e^{-px^r}\right)$$

with $\theta \equiv x^2\ d/dx$.

Further we obtain the modified moments of the weight function $x^{\lambda}\ e^{-ax^r}\ (\ln x)^w$, $w = 1,2$ on $[0,\infty)$ with respect to the product of generalized polynomials $G_n^{(\alpha)}(x,r,p,1)$. Several particular cases are obtained, some of which are known.

KEY WORDS: PRODUCT, INTEGRALS, GENERALIZED POLYNOMIALS, MODIFIED MOMENTS.

RESUMEN

En este trabajo consideramos integrales del producto de dos polinomios de la forma

$$\int_0^\infty x^\lambda e^{-ax^r} G_n^{(\alpha)}(bx, r, p, 1) \; G_m^{(\beta)}(cx, r, q, 1)\, dx$$

donde $\lambda > -1$, $r > 0$, $p+q < a$ y $G_n^{(\alpha)}(x, r, p, 1)$ es una clase de polinomios unificados definidos por la fórmula generalizada de Rodriguez

$$G_n^{(\alpha)}(x, r, p, 1) = \frac{1}{n!} x^{-\alpha-n} e^{px^r} \theta^n \left(x^\alpha e^{-px^r}\right)$$

con $\theta \equiv x^2\, d/dx$.

Además obtenemos los momentos modificados de la función de peso $x^\lambda e^{-ax^r} (\ln x)^w$, $w = 1,2$ sobre $[0,\infty)$ con respecto al producto de los polinomios generalizados $G_n^{(\alpha)}(x,r,p,1)$. Se obtienen varios casos particulares, algunos de los cuales son conocidos.

1. INTRODUCTION

Srivastava and Singhal [7] considered a unification of classes of polynomials defined by the generalized Rodrigues' formula

$$G_n^{(\alpha)}(x, r, p, k) = \frac{1}{n!} x^{-\alpha-kn} e^{px^r} \theta^n \left(x^\alpha e^{-px^r}\right) \tag{1}$$

where $\theta \equiv x^{k+1}D$, $D \equiv d/dx$.

(1) evidently provides us with an elegant generalization of the various extensions of the classical Hermite and Laguerre polynomials given by various authors ([1], [4], [5]). In particular we have:

$$G_n^{(\alpha)}(x,\ 1,\ 1,\ 1,\ k) = \frac{1}{n!}\ G_{n,k}^{(\alpha)}(x) \tag{2}$$

$$G_n^{(\alpha+n)}(x,\ r,\ 1,\ -1) = G_n^{(\alpha+1)}(x,\ r,\ 1,\ 1) = P_{n,r}^{(\alpha)}\ (x) \tag{3}$$

$$G_n^{(\alpha)}(x,\ r,\ p,\ -1) = G_n^{(\alpha-n+1)}(x,\ r,\ p,\ 1) = \frac{(-x)^n}{n!} \times$$

$$H_n^r(x,\ \alpha,\ p) \tag{4}$$

$$G_n^{(\alpha+n)}(x,\ r,\ p,\ -1) = G_n^{(\alpha+1)}(x,\ r,\ p,\ 1) = L_n^{(\alpha)}(x,\ r,\ p) \tag{5}$$

$$G^{(0)}(x,\ 2,\ 1,\ -1) = G_n^{(-n+1)}(x,\ 2,\ 1,\ 1) = \frac{(-x)^n}{n!}\ H_n(x) \tag{6}$$

$$G_n^{(\alpha+n)}(x,\ 1,\ 1,\ -1) = G_n^{(\alpha+1)}(x,\ 1,\ 1,\ 1) = L_n^{(\alpha)}(x) \tag{7}$$

where $H_n(x)$ and $L_n^{(\alpha)}(x)$ are the Hermite and Laguerre polynomials respectively.

An explicit form of (1) is the following:

$$G_n^{(\alpha)}(x,\ r,\ p,\ k) = \frac{k^n}{n!}\sum_{j=0}^{n}\frac{(-p)^j}{j!}\left(\frac{\alpha + rj}{k}\right)_n x^{rj} \tag{8}$$

which for k = 1 reduces to:

$$G_n^{(\alpha)}(x,\ r,\ p,\ 1) = \frac{r^n}{n!}\ \prod_{h=0}^{r-1}\frac{\Gamma\left(\frac{\alpha+n+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right)}\ e^{px^r} \times$$

$$_rF_r\left(\left(\frac{\alpha+n+h}{r}\right)_{h=0}^{r-1}\ ;\ \left(\frac{\alpha + h}{r}\right)_{h=0}^{r-1}\ ;\ -px^r\right) \tag{9}$$

where $_pF_q\left(\alpha_p;\ \beta_q;\ z\right)$ is the generalized hypergeometric function.

In this paper we consider integrals of the product of two polynomials of the form

$$I = \int_0^{\infty} x^{\lambda} e^{-ax^r} G_n^{(\alpha)}(bx, r, p, 1) \; G_m^{(\beta)}(cx, r, q, 1)\, dx \qquad (10)$$

$\lambda > -1$, $r > 0$, $p + q < a$

and we obtain the modified moments of the weight function $x^{\lambda} e^{-ax^r}$ $(\ln x)^w$, $w = 1,2$ on $[0,\infty)$ with respect to the product of generalized polynomials as defined in (9). Several particular cases are obtained, some of which are known.

2. MAIN RESULTS

We first consider integrals of the form:

$$I_1 = \int_0^{\infty} x^{\lambda} e^{-ax^r} G_n^{(\alpha)}(bx, r, p, 1)\, dx \qquad (11)$$

$\lambda > -1$, $r > 0$, $p < a$.

From (9) and using the serie expansion of the generalized hypergeometric function, we obtain

$$I_1 = \frac{r^n}{n!} \prod_{h=0}^{r-1} \frac{\Gamma\left(\frac{\alpha+n+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right)} \sum_{j=0}^{\infty} \prod_{h=0}^{r-1} \frac{\left(\frac{\alpha+n+h}{r}\right)_j}{\left(\frac{\alpha+h}{r}\right)_j j!} \left(-pb^r\right)^j \quad \times$$

$$\int_0^{\infty} x^{\lambda+rj} \; e^{-(a-pb^r)x^r}\, dx \qquad (12)$$

making a simple change of variable and from the definition of the gamma function, (12) can be written as:

$$I_1 = \frac{r^{n-1}}{n!} \prod_{h=0}^{r-1} \frac{\Gamma\left(\frac{\alpha+n+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right)} \frac{\Gamma\left(\frac{\lambda+1}{r}\right)}{\left(a-pb^r\right)^{(\lambda+1)/r}} \quad \times$$

$$ {}_{r+1}F_r \left(\begin{matrix} \frac{\lambda+1}{r} \; ; \; \left(\frac{\alpha+n+h}{r}\right)_{h=0}^{r-1} \\ \left(\frac{\alpha+h}{r}\right)_{h=0}^{r-1} \end{matrix} \; ; \; - \frac{pb^r}{a-pb^r} \right) \qquad (13) $$

Now, we turn out our attention to the integrals (10).

Substituting $G_m^{(\beta)}(cx, r, q, 1)$ by its series expansion given by (9), interchanging the order of integral and sum (which is justified with the given conditions) and using the result (13), we obtain

$$ I = \frac{r^{m+n-1}}{m!n!} \frac{\Gamma\left(\frac{\lambda+1}{r}\right)}{\left(a-pb^r-qc^r\right)^{(\lambda+1)/r}} \prod_{h=0}^{r-1} \frac{\Gamma\left(\frac{\alpha+n+h}{r}\right) \Gamma\left(\frac{\beta+m+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right) \Gamma\left(\frac{\beta+h}{r}\right)} \times $$

$$ \sum_{j=0}^{\infty} \prod_{h=0}^{r-1} \frac{\left(\frac{\beta+m+h}{r}\right)_j}{\left(\frac{\beta+h}{r}\right)_j j!} \left(\frac{\lambda+1}{r}\right)_j \left(- \frac{qc^r}{a-pb^r-qc^r}\right)^j \times $$

$$ {}_{r+1}F_r \left(\begin{matrix} \frac{\lambda+1}{r} +j \; , \; \left(\frac{\alpha+n+h}{r}\right)_{h=0}^{r-1} \\ \left(\frac{\alpha+h}{r}\right)_{h=0}^{r-1} \end{matrix} \; ; \; - \frac{pb^r}{a-pb^r-qc^r} \right) \qquad (14) $$

Hence,

$$ \int_0^{\infty} x^{\lambda} e^{-ax^r} G_n^{(\alpha)}(bx, r, p, 1) \; G_m^{(\beta)}(cx, r, q, 1) \, dx = $$

$$ \frac{r^{m+n-1}}{m!n!} \frac{\Gamma\left(\frac{\lambda+1}{r}\right)}{\left(a-qc^r-pb^r\right)^{(\lambda+1)/r}} \prod_{h=0}^{r-1} \frac{\Gamma\left(\frac{\alpha+n+h}{r}\right) \Gamma\left(\frac{\beta+m+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right) \Gamma\left(\frac{\beta+h}{r}\right)} \times $$

$$ F^{\,1\,:\,r\,:\,r}_{\,0\,:\,r\,:\,r} \left[\begin{matrix} \frac{\lambda+1}{r} \; : \; \left(\frac{\beta+m+h}{r}\right)_{h=0}^{r-1} \; ; \; \left(\frac{\alpha+n+h}{r}\right)_{h=0}^{r-1} \\ \text{———} \; : \; \left(\frac{\beta+h}{r}\right)_{h=0}^{r-1} \; ; \; \left(\frac{\alpha+h}{r}\right)_{h=0}^{r-1} \end{matrix} \; ; - \frac{qc^r}{a-qc^r-pb^r} \; , $$

$$- \frac{pb^r}{a-qc^r-pb^r} \Bigg] \qquad (15)$$

$$\lambda > -1,\ r > 0,\ \left|\frac{qc^r}{a-pb^r-qc^r}\right| + \left|\frac{pb^r}{a-qc^r-pb^r}\right| < 1\ ,\ p + q < a$$

where $F^{p:q:k}_{u:m:n}$ denote Kampé de Fériet's double hypergeometric function [6].

Differentiating (15) with respect to the parameter λ, we obtain,

$$\int_0^\infty x^\lambda \ln x\ e^{-ax^r}\ G_n^{(\alpha)}(bx,\ r,\ p,\ 1)\ \ G_m^{(\beta)}(cx,\ r,\ q,\ 1)\ \ dx =$$

$$\frac{r^{m+n-2}}{m!n!}\ \frac{\Gamma\left(\frac{\lambda+1}{r}\right)}{\left(a-qc^r-pb^r\right)^{(\lambda+1)/r}}\ \prod_{h=0}^{r-1} \frac{\Gamma\left(\frac{\alpha+n+h}{r}\right)\Gamma\left(\frac{\beta+m+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right)\Gamma\left(\frac{\beta+h}{r}\right)}\ \times$$

$$\sum_{j,k=0}^{\infty} \frac{\left(\frac{\lambda+1}{r}\right)_{j+k} \prod_{h=0}^{r-1}\left(\frac{\beta+m+h}{r}\right)_j \prod_{h=0}^{r-1}\left(\frac{\alpha+n+h}{r}\right)_k}{\prod_{h=0}^{r-1}\left(\frac{\beta+h}{r}\right)_j \prod_{h=0}^{r-1}\left(\frac{\alpha+h}{r}\right)_k\ j!\ k!} \left(-\frac{qc^r}{a-qc^r-pb^r}\right)^j \times$$

$$\left(-\frac{pb^r}{a-qc^r-pb^r}\right)^k \left[\Psi\left(\frac{\lambda+1}{r} + j + k\right) - \ln\left(a - qc^r - pb^r\right)\right] \qquad (16)$$

$$\lambda > -1,\ r > 0,\ p + q < a,\ \left|\frac{qc^r}{a-qc^r-pb^r}\right| + \left|\frac{pb^r}{a-qc^r-pb^r}\right| < 1.$$

Differentiating (16) with respect to λ, we have the other modified moment,

$$\int_0^\infty x^\lambda \ln^2 x\ e^{-ax^r}\ \ G_n^{(\alpha)}(bx,\ r,\ p,\ 1)\ \ G_m^{(\beta)}(cx,\ r,\ q,\ 1)\ dx =$$

$$\frac{r^{m+n-3}}{m!n!}\ \frac{\Gamma\left(\frac{\lambda+1}{r}\right)}{\left(a-qc^r-pb^r\right)^{(\lambda+1)/r}}\ \prod_{h=0}^{r-1}\frac{\Gamma\left(\frac{\alpha+n+h}{r}\right)\ \Gamma\left(\frac{\beta+m+h}{r}\right)}{\Gamma\left(\frac{\alpha+h}{r}\right)\ \Gamma\left(\frac{\beta+h}{r}\right)}\ \times$$

$$\sum_{j,k=0}^{\infty}\frac{\left(\frac{\lambda+1}{r}\right)_{j+k}\ \prod_{h=0}^{r-1}\left(\frac{\beta+m+h}{r}\right)_j\ \prod_{h=0}^{r-1}\left(\frac{\alpha+n+h}{r}\right)_k}{\prod_{h=0}^{r-1}\left(\frac{\beta+h}{r}\right)_j\ \prod_{h=0}^{r-1}\left(\frac{\alpha+h}{r}\right)_k\ j!\ \ k!}\left(-\frac{qc^r}{a-qc^r-pb^r}\right)^j\ \times$$

$$\left(-\frac{pb^r}{a-qc^r-pb^r}\right)^k\left\{\Psi'\left(\frac{\lambda+1}{r}+j+k\right)+\left[\Psi\left(\frac{\lambda+1}{r}+j+k\right)-\right.\right.$$

$$\left.\left.\ln\left(a-qc^r-pb^r\right)\right]^2\right\} \tag{17}$$

$$\lambda > -1\ ,\ r > 0\ ,\ p+q < a\ ,\ \left|\frac{qc^r}{a-qc^r-pb^r}\right| + \left|\frac{pb^r}{a-qc^r-pb^r}\right| < 1.$$

3. PARTICULAR CASES

1) If in (13) $\alpha \longrightarrow \alpha + 1$ and $r = p = 1$, we get

$$\int_0^\infty x^\lambda\ e^{-ax}\ L_n^{(\alpha)}(bx)\ dx = \frac{\Gamma(\lambda+1)}{n!}\ (\alpha+1)_n\ \ a^{-\lambda-1}\ \times$$

$${}_2F_1\left(\lambda+1,\ -n;\ \alpha+1;\ \frac{b}{a}\right) \tag{18}$$

From (18) with $a = b = 1$,

$$\int_0^\infty x^\lambda\ e^{-x}\ L_n^{(\alpha)}(x)\ dx = \frac{\Gamma(\lambda+1)}{n!}\ (-1)^n\ \frac{\Gamma(\lambda-\alpha+1)}{\Gamma(\lambda-\alpha-n+1)} \tag{19}$$

which was given by Gatteschi [2].

2) If in (15): $\alpha \longrightarrow \alpha + 1$, $\beta \longrightarrow \beta + 1$, $p = q = r = 1$,

$$\int_0^{\infty} x^{\lambda} e^{-ax} L_n^{(\alpha)}(bx) L_m^{(\beta)}(cx)\, dx = \frac{\Gamma(\lambda+1)}{(a-c-b)^{\lambda+1}\, m!n!} \times$$

$$\frac{\Gamma(\alpha+n+1)\ \Gamma(\beta+m+1)}{\Gamma(\alpha+1)\ \Gamma(\beta+1)} F^{1:1:1}_{o:1:1}\left[\begin{matrix}\lambda+1: & \beta+m+1 & ; \alpha+n+1; \\ \overline{\qquad}: & \beta+1 & ; \alpha+1 \quad ;\end{matrix} - \frac{c}{a-c-b},\right.$$

$$\left. - \frac{b}{a-c-b}\right] \qquad (20)$$

where $L_n^{(\alpha)}(x)$ are the Laguerre's polynomials.

Although [6]

$$F_2 = F^{1\ :\ 1\ :1}_{0\ :\ 1\ :1}$$

we can write (20) in the form,

$$\int_0^{\infty} x^{\lambda} e^{-ax} L_n^{(\alpha)}(bx) L_m^{(\beta)}(cx)\, dx = \frac{\Gamma(\lambda+1)}{(a-c-b)^{\lambda+1}\, m!n!} \times$$

$$\frac{\Gamma(\alpha+n+1)\ \Gamma(\beta+m+1)}{\Gamma(\alpha+1)\ \Gamma(\beta+1)} F_2\left(\lambda+1,\ \beta+m+1,\ \alpha+n+1;\ \beta+1,\ \alpha+1;\right.$$

$$\left. - \frac{c}{a-c-b},\ - \frac{b}{a-c-b}\right) \qquad (21)$$

(21) is equivalent to,

$$\int_0^{\infty} x^{\lambda} e^{-ax} L_n^{(\alpha)}(bx) L_m^{(\beta)}(cx)\, dx = \frac{(\alpha+1)_n\ (\beta+1)_m}{n!\quad m!} \frac{\Gamma(\lambda+1)}{a^{\lambda+1}} \times$$

$$F_2\left(\lambda+1,\ -m,\ -n;\ \beta+1,\ \alpha+1;\ \frac{c}{a},\ \frac{b}{a}\right) \qquad (22)$$

which concur with that given by González and Kalla [3] for ν=n and μ=m.

3) If in (16) and (17): $\alpha \longrightarrow \alpha + 1$, $\beta \longrightarrow \beta + 1$, $p = q = r = 1$,

$$\int_0^\infty x^\lambda \ln^w x \; e^{-ax} \; L_n^{(\alpha)}(bx) \; L_m^{(\beta)}(cx) \, dx = \frac{1}{m! \; n!} \frac{\Gamma(\lambda+1)}{(a-c-b)^{\lambda+1}} \times$$

$$\frac{\Gamma(\alpha+1+n) \; \Gamma(\beta+1+m)}{\Gamma(\alpha+1) \; \Gamma(\beta+1)} \sum_{j,k=0}^{\infty} \frac{(\lambda+1)_{j+k}}{(\beta+1)_j} \frac{(\beta+1+m)_j}{(\alpha+1)_k} \frac{(\alpha+1+n)_k}{j! \; k!} \left(-\frac{c}{a-c-b}\right)^j \times$$

$$\left(-\frac{b}{a-c-b}\right)^k S_w, \quad w = 1,2. \tag{23}$$

where

$S_1 = \psi(\lambda+1+j+k) - \ln(a-c-b)$, and

$S_2 = \psi'(\lambda+1+j+k) + [\psi(\lambda+1+j+k) - \ln(a-c-b)]^2$

Many other particular cases involving generalized polynomials (Laguerre,Hermite and Jacobi) can be obtained from the main results by using the relations (3),(4) and (5).

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