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ORTHOGONAL SERIES EXPANSIONS OF GENERALIZED FUNCTIONS AND THE FINITE GENERALIZED HANKEL-CLIFFORD TRANSFORMATION OF DISTRIBUTION

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ABSTRACT: In this paper the finite generalized Hankel-Clifford transformation is extended to certain spaces of (distributions) generalized functions. The operational calculus generated is used in solving certain partial differential equations, involving the generalized Kepinski-Myller-Lebedev differential operator.

KEY WORDS: Finite generalized Hankel-Clifford transformation, Bessel functions, testing function space, Schwartz distribution, operational calculus.

1 Introduction

Malgonde [2] investigated the following variant of the generalized Hankel-Clifford transform defined by

$$(h_{\alpha,\beta}f)(y)=F(y)=\int_0^\infty (y/x)^{-(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\sqrt{xy}\right)f(x)dx\ ,\ (\alpha\text{-}\beta)\geq -1/2$$
$$=y^{-\alpha-\beta}\int_0^\infty \mathcal{J}_{\alpha,\beta}(xy)f(x)dx\ ,\ (\alpha\text{-}\beta)\geq -1/2 \qquad (1.1)$$

where $\mathcal{J}_{\alpha,\beta}(x)=x^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{x})$, $J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha\text{-}\beta)$, in spaces of generalized functions.Note that (1.1) reduces to well-known Hankel-Clifford transform for suitable values of the parameters viz. for α=0 and $\beta=-\mu$, *a* transform studied in [4].

In this paper following Dorta and Méndez-Pérez [1] we establish finite version of (1.1), firstly from a classical point of view. For it we begin by considering a Fourier-Bessel type of series expansion which suggests the definition of the classical finite generalized Hankel-Clifford transformation. Later this transform is extended to certain spaces of (distributions) generalized functions.

In order to do that, we modify previously in a natural way the method developed by Zemanian [7] in his research on a variety of distributional series expansions. Recall, that the success of Zemanian's method lies in the fact that the differential operators considered are always selfadjoint.

The main objective of our work is to give a procedure that turns out to be valid for more general operators having the same positive eigenvalues and whose respective systems of eigenfunctions verify the same orthogonality condition, are simultaneously considered. Then two testing function spaces and their duals are constructed such that certain Fourier-Bessel series converge in them. So we are led on to define two distributional finite generalized Hankel-Clifford transformations, which will be called the distributional finite generalized Hankel-Clifford transformations of the first kind of order (α–β). This approach is reminiscent of the procedure described in [3 and 4] for extending the infinite generalized Hankel-Clifford transformation to distributional spaces. Finally, the operational calculus generated is used in solving directly certain partial differential equations, involving the generalized Kepinski-Myller-Lebedev differential operator.

2 Preliminary Results

Considering the Sturm-Liouville problem

$$\left(\Delta_{\alpha,\beta}+\lambda^2\right)y=0, a\le x\le b, \tag{2.1}$$

$$a_1y(a)+a_2y'(a)=0, b_1y(b)+b_2y'(b)=0 \tag{2.2}$$

where $\Delta_{\alpha,\beta}=x^{\beta}Dx^{\alpha-\beta+1}Dx^{-\alpha}$; a,a_1,a_2,b,b_1,b_2 are real constants and $D=\dfrac{d}{dx}$.

The general solution of the equation (2.1) is

$$y=\phi_\lambda(x)=A(\lambda)\mathcal{J}_{\alpha,\beta}(\lambda x)+B(\lambda)\Upsilon_{\alpha,\beta}(\lambda x)$$

$$or\, y=\phi_{\lambda_n}(x)=A(\lambda_n)\mathcal{J}_{\alpha,\beta}(\lambda_n x)+B(\lambda_n)\Upsilon_{\alpha,\beta}(\lambda_n x) \tag{2.3}$$

where $\mathcal{J}_{\alpha,\beta}(\lambda_n x)=(x\lambda_n)^{\left(\frac{\alpha+\beta}{2}\right)} J_{\alpha-\beta}(2\sqrt{x\lambda_n}),\Upsilon_{\alpha,\beta}(\lambda_n x)=(x\lambda_n)^{\left(\frac{\alpha+\beta}{2}\right)} Y_{\alpha-\beta}(2\sqrt{x\lambda_n})$

and $Y_{\alpha-\beta}(z)$ is the Bessel – function of the second kind of order $(\alpha-\beta)$.

Equation (2.1) may be written as

$$x^{(\alpha+\beta)}\frac{d}{dx}(x^{1-(\alpha+\beta)}y')^2+\left[\lambda^2 x^{1-(\alpha+\beta)}+\frac{\alpha\beta}{x^{(\alpha+\beta)}}\right]\frac{d}{dx}\left(y^2\right)=0$$

On integrating by parts in the interval $[a,b]$, we obtain

$$(1-\alpha-\beta)\lambda^2\int_a^b x^{-\alpha-\beta}y^2dx-\left[x^{2-(\alpha+\beta)}y'^2+\lambda^2x^{1-(\alpha+\beta)}y^2+\frac{\alpha\beta}{x^{(\alpha+\beta)}}y^2\right]_a^b$$
$$-\int_a^b(\alpha+\beta)x^{1-(\alpha+\beta)}y'^2dx+(\alpha+\beta)\alpha\beta\int_a^b x^{-(\alpha+\beta)-1}y^2dx=0 \qquad (2.4)$$

Let $y=\phi_n(x)$ be the eigenfunctions of the problem (2.1)–(2.2), which correspond to the non-zero eigen-values λ_n .Then the orthogonality condition

$$\int_a^b x^{-(\alpha+\beta)}\phi_m(x)\phi_n(x)dx$$
$$=\frac{1}{\lambda_n^{\ 2}}\left[\begin{array}{l}x^{2-(\alpha+\beta)}\{\frac{d}{dx}(\phi_n(x))\}^2+\lambda_n^{\ 2}x^{1-(\alpha+\beta)}(\phi_n(x))^2\\ +\alpha\beta x^{-(\alpha+\beta)}(\phi_n(x))^2-(\alpha+\beta)x^{1-(\alpha+\beta)}\phi_n(x)[\frac{d}{dx}(\phi_n(x)]\end{array}\right]_a^b$$

if $m=n$

$=0$, if $m\neq n$, (2.5)

may be derived from (2.4) and Sturm-Liouville general theory.

Consider now the problem

$\left(\Delta_{\alpha,\beta}+\lambda^2\right)\phi(x)=0\,,0\le x\le a\,,$ (2.6)

$\phi(a)=0$

whose solution is in view of (2.3)

$\phi_n(x)=\mathcal{J}_{\alpha,\beta}(\lambda_n x)$ (2.7)

where $\lambda_1,\lambda_2,...$ represent the positive zeros arranged in ascending order of magnitude of the equation [6,p. 479]

$$\mathcal{J}_{\alpha,\beta}(\lambda_n a) = 0 \tag{2.8}$$

The above orthogonality condition (2.5) now becomes

$$= \int_0^a x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_m x) \mathcal{J}_{\alpha,\beta}(\lambda_n x) dx$$

$$= \begin{cases} a^{2-\alpha-\beta} \lambda_n \mathcal{J}^{z}_{\alpha.\beta-1}(\lambda_n a); m = n \\ 0 \qquad ; m \neq n \end{cases} \tag{2.9}$$

3 The Fourier-Bessel Series and the classical finite generalized Hankel-Clifford transformation

Let $f(x)$ be an arbitrary function defined in (0, a).Then the Fourier -Bessel type series expansion can be expressed by virtue of (2.9) as the following Fourier Bessel expansion

$$f(x) = \sum_{m=1}^{\infty} a_m \mathcal{J}_{\alpha,\beta}(\lambda_m x) \tag{3.1}$$

where $a_m = \dfrac{1}{a^{2-\alpha-\beta} \lambda_m \mathcal{J}^{z}_{\alpha.\beta-1}(\lambda_m a)} \displaystyle\int_0^a x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_m x) f(x) dx$ (3.2)

The convergence of the series (3.1) is straightforward and the following Theorem 3.1 follows from [1].

<u>THEOREM</u> 3.1. Let $f(x)$ be a function defined and absolutely integrable on (0, a).

Assume that $(\alpha - \beta) \geq -\dfrac{1}{2}$ and set

$$a_m = \frac{1}{a^{2-\alpha-\beta} \lambda_m \mathcal{J}^{z}_{\alpha.\beta-1}(\lambda_m a)} \int_0^a t^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_m t) f(t) dt, m = 1,2,3...$$

If $f(t)$ is of the bounded variation in $(a_1, a_2), (0 < a_1 < a_2 < a)$ and if $t \in (a_1, a_2)$, then the series

$\sum_{m=1}^{n} a_m \mathcal{J}_{\alpha,\beta}(\lambda_m x)$ converges to $\dfrac{1}{2}[f(x+0) + f(x-0)]$.

Expression (3.1) and (3.2) and the Theorem 3.1 suggest to introduce the integral transform

$$(\hbar_{1,\alpha,\beta}f)(n) = (\hbar_{\alpha,\beta}f)(n) = F_{\alpha,\beta}(n) = \int_0^a x^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(\lambda_n x) f(x) dx \tag{3.3}$$

which will be called the finite generalized Hankel-Clifford transformation of the first kind.

Its inversion formula is given by

$$(\hbar^{-1}{}_{\alpha,\beta}F_{\alpha,\beta})(x) = f(x) = \frac{1}{a^{2-\alpha-\beta}} \sum_{n=1}^{\infty} \frac{F_{\alpha,\beta}(n)\mathcal{J}_{\alpha,\beta}(\lambda_n x)}{\lambda_n \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)} \tag{3.4}$$

We point out the following operational rules:

1) If $f(x) \in C^2(0,a)$, upon integrating by parts we deduce the relation

$$\hbar_{\alpha,\beta}[xf''(x) + (1-\alpha-\beta)f'(x) + \alpha\beta x^{-1} f(x)]$$
$$= f(a)\lambda_n \, a^{(1-\alpha-\beta)} \mathcal{J}_{\alpha,\beta-1}(\lambda_n a) - \lambda_n \hbar_{\alpha,\beta}[f(x)]$$

2) If $f(x) \in C^{2r}(0,a), f^{(i)}(0)$ are finite and $f^{(i)}(a) = 0; i = (1,2,3,...2r-2)$, then

$$\hbar_{\alpha,\beta}[xf''(x) + (1-\alpha-\beta)f'(x) + \alpha\beta x^{-1} f(x)]^r = (-1)^r \lambda_n^r \, \hbar_{\alpha,\beta}[f(x)],$$

r being a positive integer.

4 The testing function spaces $A_{\alpha,\beta}$ and $A^*_{\alpha,\beta}$ and their duals

In this section we shall employ the same notation and terminology as those used in Zemanian [7].

Thus I denote the interval (0, a) and $(\alpha-\beta)$ will be restricted to the interval $-\frac{1}{2} \le (\alpha-\beta) < \infty$. $L_2(I)$ and $L^*_2(I)$ represent the spaces of equivalence classes of functions that are quadratically integrable on *I* with regard to the weight functions $x^{-(\alpha+\beta)}$ and $x^{(\alpha+\beta)}$. A mixed inner product is defined on $L_2(I) \times L^*_2(I)$ by

$$(f,g) = \int_I f(x)\overline{g(x)}dx, f \in L_2(I), g \in L^*_2(I) \tag{4.1}$$

where $\overline{g(x)}$ denotes the complex conjugate of *g(x)*. This definition is consistent with the inner product considered usually on $L_2(I)$ and $L^*_2(I)$. Indeed,(4.1) can be rewritten as

$$(f,g) = \int_I x^{-(\alpha+\beta)} f(x)(x^{(\alpha+\beta)}\overline{g(x)})dx = \int_I x^{(\alpha+\beta)}(x^{-(\alpha+\beta)}f(x))\overline{g(x)}dx$$

and note that $f(x)$ and $x^{(\alpha+\beta)}g(x)$ belong to $L_2(I)$, while $x^{-(\alpha+\beta)}f(x)$ and $g(x)$ are in $L_2^*(I)$. $D(I), E(I), D'(I)$ and $E'(I)$ are well-known spaces of testing functions and their duals [8, p. 32].

The differential operator

$$\Delta_{\alpha,\beta} = x^{\beta}Dx^{\alpha-\beta+1}Dx^{-\alpha} = xD^2 + (1-\alpha-\beta)D + x^{-1}\alpha\beta \tag{4.2}$$

is not self-adjoint. We consider, together with $\Delta_{\alpha,\beta}$, the operator

$$\Delta^*_{\alpha,\beta} = x^{-\alpha}Dx^{\alpha-\beta+1}Dx^{\beta} = xD^2 + (1+\alpha+\beta)D + x^{-1}\alpha\beta \tag{4.3}$$

$\Delta^*_{\alpha,\beta}$ is called the adjoint operator of $\Delta_{\alpha,\beta}$.

Note that $\Delta^*_{\alpha,\beta} = x^{-\alpha-\beta}\Delta_{\alpha,\beta}x^{\alpha+\beta}$ (4.4)

The functions

$$\phi_n(x) = \mathcal{G}_{\alpha,\beta}(\lambda_n x) \quad ,n = 1,2,..... \tag{4.5}$$

are the eigenfunctions of $\Delta_{\alpha,\beta}$,whereas the functions

$$\phi^*_n(x) = x^{-(\alpha+\beta)}\mathcal{G}_{\alpha,\beta}(\lambda_n \mathrm{x}) \quad ,n = 1,2,..... \tag{4.6}$$

are the eigenfunctions of $\Delta^*_{\alpha,\beta}$.

In both the cases we have the same eigenvalues of λ_n (n=1,2,…) which are the positive roots of equation (2.8).Therefore,

$$(\Delta_{\alpha,\beta} + \lambda_n^{\ 2})\phi_n(x) = 0$$

and (4.7)

$$(\Delta^*_{\alpha,\beta} + \lambda_n^{\ 2})\phi^*_n(x) = 0$$

Systems of eigenfunctions $\{\phi_n(x)\}_{n=1}^{\infty}$ and $\{\phi_n^{\ *}(x)\}_{n=1}^{\infty}$ verify by (2.9) and (4.1) the orthogonality condition

$$\left(\phi_n, \phi^*_m\right) = \left(\phi^*_m, \phi_n\right) = \begin{cases} a^{2-\alpha-\beta}\lambda_n\,\mathcal{G}^2_{\alpha,\beta-1}(\lambda_n a) \text{ ,} if \;\; m = n \\ 0 \qquad\qquad\qquad\qquad \text{, if } \; m \neq n \end{cases} \tag{4.8}$$

This is equivalent to say that $\{\phi_n(x)\}_{n=1}^{\infty}$ is orthogonal with respect to the weight function $x^{-(\alpha+\beta)}$ and, on the other hand, that $\{\phi^*_n(x)\}_{n=1}^{\infty}$ is orthogonal with respect to $x^{(\alpha+\beta)}$. In any case (4.8) holds.

$A_{\alpha,\beta}$ is defined as the linear space of all infinitely differentiable complex-valued functions $\phi(x)$ on I such that

(i) $\varsigma_{k,\alpha,\beta}\phi(x) = \left[\int_I x^{(\alpha+\beta)} \left| \Delta^{*^k}{}_{\alpha,\beta}\phi(x) \right|^2 dx \right]^{\frac{1}{2}}$ exists for every k=0,1,2,...

(ii) $\left(\Delta^{*^K}{}_{\alpha,\beta}\phi, \phi_n \right) = \left(\phi, \Delta^k{}_{\alpha,\beta}\phi_n \right)$ holds for each n and k.

$A_{\alpha,\beta}$ is the countable multinormed space having the topology generated by $\{\varsigma_{\alpha,\beta}\}$.

$A_{\alpha,\beta}$ is also complete. Consequently, $A_{\alpha,\beta}$ is a Fréchet space.

In our context we can establish a result analogous to [7, Lemma 1]

THEOREM 4.1. Let $(\alpha-\beta) \geq -\frac{1}{2}$. Every member $\phi \in A_{\alpha,\beta}$ can be expanded into a generalized series of the form

$$\phi = \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta}\lambda_n\, \vartheta^2_{\alpha,\beta-1}(\lambda_n a)} (\phi,\phi_n)\phi_n^*, \qquad (4.9)$$

which converges in $A_{\alpha,\beta}$.

Proof. . Note that $\Delta^{*k}{}_{\alpha,\beta}\phi \in L^*_2(I)$. Hence by (ii) and (4.7) we have

$$\Delta^{*k}{}_{\alpha,\beta}\phi = \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta}\lambda_n\, \vartheta^2_{\alpha,\beta-1}(\lambda_n a)} \left(\Delta^{*k}{}_{\alpha,\beta}\phi,\phi_n\right)\phi_n^*$$

$$= \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta}\lambda_n\, \vartheta^2_{\alpha,\beta-1}(\lambda_n a)} \left(\phi,\Delta^{k}{}_{\alpha,\beta}\phi_n\right)\phi_n^*$$

$$= \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta}\lambda_n\, \vartheta^2_{\alpha,\beta-1}(\lambda_n a)} (\phi,\phi_n)\Delta^{*k}{}_{\alpha,\beta}\phi_n^*$$

where the series involved converge in $L^*{}_2(I)$ .Therefore

$$\varsigma_{k,\alpha,\beta}\left[\phi - \sum_{n=1}^{\infty} \frac{1}{a^{2-\alpha-\beta}\lambda_n\, \vartheta^2_{\alpha,\beta-1}(\lambda_n a)} (\phi,\phi_n)\phi_n^* \right] \to 0$$

as n$\to\infty$. This completes the proof of Theorem 4.1.

$A'_{\alpha,\beta}$ is the dual space of $A_{\alpha,\beta}$ and too complete. We now list some of the properties of these spaces

(a) $D(I) \subset A_{\alpha,\beta} \subset E(I))$. $E'(I)$ is a subspace of $A'_{\alpha,\beta}$.

(b) It can be seen that $\phi_n^*(x)$, given by (4.6), belongs to $A_{\alpha,\beta}$.

(c) The operation $\phi \to \Delta^{*k}{}_{\alpha,\beta}\phi$ is a continuous linear mapping of $A_{\alpha,\beta}$ into itself.

Consequently, the operation $f \to \Delta_{\alpha,\beta} f$ defined on $A'_{\alpha,\beta}$ by

$$\left(\Delta_{\alpha,\beta} f, \phi\right) = \left(f, \Delta^*{}_{\alpha,\beta}\phi\right) \tag{4.10}$$

is also a continuous linear mapping of $A'_{\alpha,\beta}$ into itself.

We will have need of another testing function space $A^*{}_{\alpha,\beta}$ along this work. $A^*{}_{\alpha,\beta}$ consists of all infinitely differentiable functions $\phi(x)$ defined on I such that

(i′) $\varsigma^*{}_{k,\alpha,\beta}\phi(x) = \left[\int_I x^{-(\alpha+\beta)} \left|\Delta^k{}_{\alpha,\beta}\phi(x)\right|^2 dx\right]^{\frac{1}{2}}$ exists for every k=0,1,2,...

(ii′) $\left(\Delta^k{}_{\alpha,\beta}\phi, \phi^*{}_n\right) = \left(\phi, \Delta^{*k}{}_{\alpha,\beta}\phi^*{}_n\right)$ holds for each n and k

As before, $A^*{}_{\alpha,\beta}$ is a Fréchet space. $A^{*\prime}{}_{\alpha,\beta}$ represents the dual space of $A^*{}_{\alpha,\beta}$.

Some properties related to these spaces are listed below

(a′) $D(I) \subset A^*{}_{\alpha,\beta} \subset E(I)$. $E'(I)$ is a subspace of $A^{*\prime}{}_{\alpha,\beta}$

(b′) Note that $\phi_n(x)$, given by (4.5), is now in $A^*{}_{\alpha,\beta}$

(c′) The operation $\phi \to \Delta_{\alpha,\beta}\phi$ is a continuous linear mapping of $A^*{}_{\alpha,\beta}$ into itself.

Hence the operation $f \to \Delta^*{}_{\alpha,\beta} f$ defined on $A^{*\prime}{}_{\alpha,\beta}$ by $\left(\Delta^*{}_{\alpha,\beta} f, \phi\right) = \left(f, \Delta_{\alpha,\beta}\phi\right)$ for any $\phi \in A^*{}_{\alpha,\beta}$ is a continuous linear mapping of $A^{*\prime}{}_{\alpha,\beta}$ into itself.

Remark 1: Since $\{\varphi^*{}_n\}$ is a orthogonal system on I with respect to the weight function $x^{(\alpha+\beta)}$, verifying the same orthogonality condition (2.9), we propose to consider this other finite generalized Hankel-Clifford transformation

$$(\hbar^*{}_{\alpha,\beta} f)(n) = F^*{}_{\alpha,\beta}(n) = \int_0^a x^{(\alpha+\beta)}\phi^*{}_n(x) f(x) dx = \int_0^a \vartheta_{\alpha,\beta}(\lambda_n x) f(x) dx \tag{4.11}$$

the inversion formula being given through

$$(\hbar^{*-1}{}_{\alpha,\beta}F^{*}{}_{\alpha,\beta})(x)=f(x)=\sum_{n=1}^{\infty}\frac{F^{*}{}_{\alpha,\beta}(n)x^{-(\alpha+\beta)}\vartheta_{\alpha,\beta}(\lambda_n x)}{a^{2-\alpha-\beta}\lambda_n\,\vartheta^2_{\alpha,\beta-1}(\lambda_n a)} \tag{4.12}$$

By using a similar reasoning as in the proof of Theorem 4.1, we can establish

THEOREM 4.2. Let $(\alpha-\beta)\geq-\frac{1}{2}$. If $\phi\in \mathrm{A}^{*}{}_{\alpha,\beta}$, then

$$\phi=\sum_{n=1}^{\infty}\frac{1}{a^{2-\alpha-\beta}\lambda_n\,\vartheta^2_{\alpha,\beta-1}(\lambda_n a)}\left(\phi,\phi^{*}_{\alpha,\beta}\right)\phi_n \tag{4.13}$$

where the series converges in $\mathrm{A}^{*}{}_{\alpha,\beta}$.

Remark 2: Observe that $(\hbar_{\alpha,\beta}\phi)(x)=\phi_{\alpha,\beta}(n)=(\phi,\phi^{*}_{\alpha,\beta}),\ \phi\in \mathrm{A}^{*}{}_{\alpha,\beta}$, (4.14)

is the finite generalized Hankel-Clifford transformation (3.3) acting on the space $\mathrm{A}^{*}{}_{\alpha,\beta}$. Thus Theorem 4.2 can be interpreted as the inversion Theorem 3.1 for all testing function $\phi\in \mathrm{A}^{*}{}_{\alpha,\beta}$. Analogously,

$$(\hbar^{*}{}_{\alpha,\beta}\phi)(n)=\phi^{*}_{\alpha,\beta}(n)=(\phi,\phi_n),\phi\in \mathrm{A}_{\alpha,\beta}, \tag{4.15}$$

can be considered as the finite generalized Hankel-Clifford transformation (4.11). Its inversion formula is given in the space $\mathrm{A}_{\alpha,\beta}$ by (4.12).

Remark 3: Assume that $(\alpha-\beta)\geq-\frac{1}{2}$. Then $\mathrm{A}_{\alpha,\beta}$ may be identified with a subspace of $\mathrm{A}^{*\prime}{}_{\alpha,\beta}$, that is, $\mathrm{A}_{\alpha,\beta}\subset \mathrm{A}^{*\prime}{}_{\alpha,\beta}$. Indeed, every member $f\in \mathrm{A}_{\alpha,\beta}$ generates a regular distribution in $\mathrm{A}^{*\prime}{}_{\alpha,\beta}$ by $(f,\phi)=\int_I f(x)\phi(x)dx,\phi\in \mathrm{A}^{*}{}_{\alpha,\beta}$, since

$$\left|(f,\phi)\right|\leq\varsigma_{0,\alpha,\beta}(f)\cdot\varsigma^{*}_{0,\alpha,\beta}(\phi).$$

Furthermore, two members of $\mathrm{A}_{\alpha,\beta}$ which give rise to the same member of $\mathrm{A}^{*\prime}{}_{\alpha,\beta}$ must be identical.

In a similar way $\mathrm{A}^{*}{}_{\alpha,\beta}$ can be considered as a subspace of $\mathrm{A}'_{\alpha,\beta}$.

5 Orthogonal series expansions of generalized functions and the distributional finite generalized Hankel-Clifford transformation

The main result of this section can be stated as follows:

THEOREM 5.1. Let $(\alpha-\beta)\geq -\frac{1}{2}$. Every member $f\in A'_{\alpha,\beta}$ can be expanded into a generalized series of the form

$$f=\sum_{n=1}^{\infty}\frac{1}{a^{2-\alpha-\beta}\lambda_n\, \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)}\left(f,\phi_n^*\right)\phi_n \tag{5.1}$$

which converges in $A'_{\alpha,\beta}$.

Proof. By virtue of Theorem 4.1 it is inferred that

$$\begin{aligned}(f,\phi)&=(f,\sum_{n=1}^{\infty}\frac{1}{a^{2-\alpha-\beta}\lambda_n\, \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)}(\phi,\phi_n^*)\phi_n^*)\\ &=\sum_{n=1}^{\infty}\frac{1}{a^{2-\alpha-\beta}\lambda_n\, \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)}\left(f,\phi_n^*\right)\overline{(\phi,\phi_n)}\\ &=\sum_{n=1}^{\infty}\frac{1}{a^{2-\alpha-\beta}\lambda_n\, \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)}\left(f,\phi_n^*\right)(\phi_n,\phi)\end{aligned}$$

for all $\phi\in A_{\alpha,\beta}$. This implies (5.1) truly converges in $A'_{\alpha,\beta}$.

Through an argument similar we can also assert

THEOREM 5.2. Let $\left(\alpha-\beta\right)\geq -\frac{1}{2}$. If $f\in A^{*\prime}_{\alpha,\beta}$, then

$$f=\sum_{n=1}^{\infty}\frac{1}{a^{2-\alpha-\beta}\lambda_n\, \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)}\left(f,\phi_n\right)\phi_n^* \tag{5.2}$$

where the series converges in $A^{*\prime}_{\alpha,\beta}$.

In the view of (5.1), the distributional finite generalized Hankel-Clifford transformation of the first kind of $f\in A'_{\alpha,\beta}$ is defined by

$$(\hbar'_{\alpha,\beta}f)(n)=F_{\alpha,\beta}(n)=\left(f(x),\phi_n^*(x)\right)=\left(f(x),x^{-(\alpha+\beta)}\mathcal{J}_{\alpha,\beta}\left(\lambda_n x\right)\right) \tag{5.3}$$

for each value of n=1,2,3,....Observe that this definition has a sense by virtue of note (b) in section 4. Its corresponding inversion formula is supplied by Theorem 5.1 and can be expressed as

$$(\hbar'^{-1}_{\alpha,\beta}F_{\alpha,\beta})(x)=f(x)=\sum_{n=1}^{\infty}\frac{F_{\alpha,\beta}(n)\mathcal{J}_{\alpha,\beta}(\lambda_n x)}{a^{2-\alpha-\beta}\lambda_n\, \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)} \tag{5.4}$$

We need merely invoke (4.10) to get

$\left(\Delta^{k}{}_{\alpha,\beta}f,\phi\right)=\left(f,\ \Delta^{*k}{}_{\alpha,\beta}\phi\right)$

for all $\phi\in A_{\alpha,\beta}$ and $k=0,1,2,....$

If ϕ is replaced by ϕ_n^{*}, and (4.7) is used, we yield

$$\left(\Delta^{k}{}_{\alpha,\beta}f,x^{-(\alpha+\beta)}\mathcal{G}_{\alpha,\beta}(\lambda_n x)\right)=\left(f,(-\lambda_n)^{k}\,x^{-(\alpha+\beta)}\mathcal{G}_{\alpha,\beta}(\lambda_n x)\right)$$

This formula may be rewritten in accordance with (5.3) as

$$\hbar'_{\alpha,\beta}\left(\Delta^{k}{}_{\alpha,\beta}f\right)=(-\lambda_n)^{k}\ \hbar'_{\alpha,\beta}f \qquad (5.5)$$

for every $f\in A'_{\alpha,\beta}$ and k=0,1,2,...

Remark 4: Theorem 5.2 suggests to introduce other variant of the distributional finite generalized Hankel-Clifford transformation of the first kind in the space $A'_{\alpha,\beta}$ by means of

$$(\hbar^{*/}_{\alpha,\beta}f)(n)=F^{*}{}_{\alpha,\beta}(n)=(f,\phi_n)=(f,x^{-(\alpha+\beta)}\mathcal{G}_{\alpha,\beta}(\lambda_n x)) \qquad (5.6)$$

where $f\in A^{*\prime}{}_{\alpha,\beta}$ for each value of n= 1,2,... The inversion formula is given through

$$(\hbar^{*/-1}_{\alpha,\beta}F^{*}{}_{\alpha,\beta})(x)=f(x)=\sum_{n=1}^{\infty}\frac{F^{*}{}_{\alpha,\beta}(n)}{a^{2-\alpha-\beta}\lambda_n\,\mathcal{G}^{2}_{\alpha,\beta-1}(\lambda_n a)}x^{-(\alpha+\beta)}\mathcal{G}_{\alpha,\beta}(\lambda_n x) \qquad (5.7)$$

Remark 5: Since $A^{*}{}_{\alpha,\beta}\subset A'_{\alpha,\beta}$, our classical finite generalized Hankel-Clifford transformation (4.14) is a special case of the generalized (distributional) transformation of (5.3) and Theorem 5.1 turns out to be an extension to distributions of Theorem 4.2. Similarly, as an immediate consequence of the inclusion $A_{\alpha,\beta}\subset A^{*\prime}{}_{\alpha,\beta}$, the classical finite generalized Hankel-Clifford transformation (4.15) agrees with the distributional finite generalized Hankel-Clifford transformation (5.6), so that Theorem 5.2 appears now as the distributional version of Theorem 4.1.

Remark 6: Let N be a linear differential operator and denote by N^{*} its adjoint operator. Zemanian [7, p.264] investigated only the case $N=N^{*}$. However, the method developed here allows to tackle more general problems (e.g., the case of our operators $\Delta_{\alpha,\beta}$ and $\Delta^{*}{}_{\alpha,\beta}$) provided that, of course, both operators have the same eigenvalues and their respective systems of eigenfunctions verify also an identical orthogonality condition with respect to suitable weight functions.

6. Application

To illustrate the use of the distributional finite generalized Hankel-Clifford transformation, we wish to solve the following generalized Kepinski-Myller-Lebedev partial differential equation, of course, now in a finite interval.

$$x\frac{\partial^2 v}{\partial x^2}+(1-\alpha-\beta)\frac{\partial v}{\partial x}-x^{-1}\alpha\beta v-\mu\frac{\partial v}{\partial t}=0, 0<x<a, t>0 \tag{6.1}$$

satisfying boundary conditions

i) As $t\to 0+$, $v(x,t)\to f(x)\in A'_{\alpha,\beta}$

ii) As $t\to\infty$, $v(x,t)$ converges uniformly to zero on $0<x<a$

iii) As $x\to a-$, $v(x,t)$ converges to zero on $t_0\le t<\infty$ for each $t_0<0$

iv) As $x\to 0+$, $v(x,t)=O(x^{(\alpha-\beta)})$ on $t_0\le t<\infty$

Let us denote $V(n,t)=\hbar_{\alpha,\beta}(v(x,t))$. According to (4.2), (6.1) becomes

$$\Delta_{\alpha,\beta}v-\mu\frac{\partial v}{\partial t}=0 \tag{6.2}$$

By applying $\hbar_{\alpha,\beta}$ to (6.2) and making use of (5.5) we arrive at

$$-\lambda_n\ V(n,t)-\mu\frac{\partial}{\partial t}V(n,t)=0,$$

whose solution is

$V(n,t)=F_{\alpha,\beta}(n)e^{-\frac{\lambda_n t}{\mu}}$, because of the boundary conditions (i) and (ii).

Here $\hbar_{\alpha,\beta}f=F_{\alpha,\beta}(n)$ and λ_n represents the n^{th} positive zero of the equation $\mathcal{J}_{\alpha,\beta}(\lambda_n a)=0$. We may now invoke the inversion formula (5.4) to provide the required

solution $v(x,t)=(\hbar^{-1}{}_{\alpha,\beta}F_{\alpha,\beta})(x)=f(x)=\sum_{n=1}^{\infty}\frac{F_{\alpha,\beta}(n)\mathcal{J}_{\alpha,\beta}(\lambda_n x)}{a^{2-\alpha-\beta}\lambda_n\ \mathcal{J}^2_{\alpha,\beta-1}(\lambda_n a)}$ (6.3)

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