Rev. Acad. Canar. Cienc., XII (Núms. 1-2), 135-146 (2000) - (Publicado en julio de 2001)

# THE EFFECT OF EXTERNAL NOISE ON THE DYNAMICS OF SPECULATIVE MARKETS

JOSÉ M. PACHECO, JOSÉ M. LÓPEZ AND CÉSAR RODRÍGUEZ

ABSTRACT. A general model for asset price dynamics in speculative markets is considered. The choice of a chartist demand function allows to check out the agreement between analytical predictions and numerical simulations of this model. A discussion is presented on the effect of market noise on speculative behaviour. First, by solving a particular Fokker-Planck equation we show that the white noise has only a disorganizing effect around the deterministic equilibrium state. Second, a separation condition is used to deduce the existence of limit cycles in the slow-fast dynamics for small values of the average time needed for the chartist formation of the price trend. Numerical results show that the effect of noise can double the wavelength of alternate slow and fast transitions.

## 1. INTRODUCTION

Several categories of models have been assessed with the aim of analyzing cycle-like behaviour in economic theory. In particular, prediction of asset market prices is an area where these models are broadly in use and there is active research where various mathematical techniques are employed. Among these dynamical systems are known to be essential tools in economic analysis, both in the continuous and in the discrete cases.

In this paper emphasis is put on the investigation of sudden changes in the behaviour of market asset prices. These changes are idealized as the onset of a slow-fast cyclic dynamics for some parameter value. In turn, cyclic behaviour itself appears as a result of a Hopf bifurcation. But in order to mimic real market behaviour, this is still too rigid a frame and dynamical systems are extended to other categories of equations.

The importance of random walks in many classical economic assumptions is the starting point for the use of the theory of stochastic differential equations in the modelling of economic problems, though alternative approaches, as reaction-diffusion equations have also been applied (see [1]). As a rule, an adequate deterministic dynamical system is built and analyzed in phase space, afterwards it is perturbed with noise or diffusion and studied (or solved) in this new setting. Usually the stochastic models are treated numerically, because for higher dimensional systems the theoretical Fokker-Planck equation approach is cumbersome and applicable only in a few cases with specific side conditions. This is the way followed in this study.

## 2. A MODEL FOR AN SPECULATIVE MARKET

Several drawbacks in the application of random walk theory and the paradigm of efficients markets led Beja and Goldmann [2] and Chiarella [3] to build more realistic models of the price evolution of an asset in a speculative market. The excess demand function is defined as the time derivative $P'(t)$, where $P$ stands for the logarithm of price. The basic hypothesis is that the excess demand function can be split into two components: a fundamental one $D(t)$ and a speculative one $d(t)$ which correspond to the two basic types of agents acting in the market:

$$P'(t) = D(t) + d(t)$$

The two types are called respectively "fundamentalists" and "chartists". The first ones make their decisions on a theoretical basis (rational expectatives), while chartists (also known as speculators) estimate prices on the basis of past price trends (adaptive expectatives). Chiarella [3] defines the fundamentalist excess demand to be proportional to the difference between the equilibrium (walrasian) price $W(t)$ in an ideal market and the actual price $P(t)$:

$$D(t) = a[W(t) - P(t)]$$

where $a > 0$ is the slope of the fundamentalist demand. On the other hand, the chartist excess demand is represented by a nonlinear function of the difference between an average estimation $\psi(t)$ of the actual trend $P'(t)$ of $P(t)$ and the yield $g(t)$ of some other less risky alternative reference, ( e.g. bonds):

$$d(t) = h(\psi(t) - g(t))$$

where $h$ is some bounded (both above and below) increasing function with a single inflection and such that $h(0) = 0$ (see [3]). The model is completed by specifying how $\psi(t)$ is built. Here adaptive expectatives are used, with the following equation defining $\psi(t)$:

$$\psi'(t) = c[P'(t) - \psi(t)]$$

where $c > 0$ is a measure of how quickly chartists adjust their offers. Its inverse $\frac{1}{c}$ can be considered as the time lag $\tau$ needed for building expectatives and will play an important role in the sequel. Therefore the model reads, dropping the $t$ dependence:

$$\begin{aligned} P' &= a[W - P] + h(\psi - g) \\ \tau\psi' &= a[W - P] - \psi + h(\psi - g) \end{aligned}$$

This differential system has a single equilibrium point $(P_e, \psi_e) = (W - \frac{h(-g)}{a}, 0)$, and under the hypothesis that $W$ and $g$ be constant a change of origin to this point yields the new system

$$\begin{aligned} p' &= -ap + k(\psi) \\ \tau\psi' &= -ap - \psi + k(\psi) \end{aligned}$$

where $p = P - (W - \frac{h(-g)}{a})$ and $k(\psi) = h(\psi - g) - h(-g)$.

A linear stability analysis can be carried on in order to show how the qualitative behaviour of the equilibrium point depends on the parameter $\tau$. The jacobian for this system is

$$\begin{pmatrix} -a & h'(\psi - g) \\ \frac{-a}{\tau} & \frac{h'(\psi-g)-1}{\tau} \end{pmatrix}$$

and at the equilibrium point it becomes

$$A = \begin{pmatrix} -a & h'(-g) \\ \frac{-a}{\tau} & \frac{h'(-g)-1}{\tau} \end{pmatrix} = \begin{pmatrix} -a & b \\ \frac{-a}{\tau} & \frac{b-1}{\tau} \end{pmatrix}$$

where $b = h'(-g) = k'(0) > 0$, this last inequality being true because of the hypotheses made on the function $h$. The quadratic equation for the eigenvalues is

$$\lambda^2 - Tr(A)\lambda + \det(A) = \lambda^2 + (a - \frac{b-1}{\tau})\lambda + \frac{ab}{\tau} - \frac{a(b-1)}{\tau} = 0$$

Therefore, whenever $a - \frac{(b-1)}{\tau} < 0$, i.e. if $\tau > \tau^* = \frac{b-1}{a}$, one has a stable equilibrium. If the reverse inequality holds, an unstable equilibrium appears. Under application of the Hopf bifurcation theorem, Chiarella [3] showed that a limit cycle exists when $\tau$ crosses the critical value $\tau^*$.

The features of the cycle can be determined with a little effort. In the last system $p$ can be eliminated in the following way. First, substitute the first equation in the second one to obtain

$$\tau\psi' = p' - \psi$$

Now, taking the time derivative in this equation:

$$\tau\psi'' = p'' - \psi'$$

but $p'' = -ap' + k'(\psi)\psi'$ from the first equation, so one has

$$\tau\psi'' = -ap' + k'(\psi)\psi' - \psi' = -a[-ap + k(\psi)] + k'(\psi)\psi' - \psi'$$

and on taking $-ap = \tau\psi' - k(\psi) + \psi$ from the second equation it follows that

$$\tau\psi'' = -a[\tau\psi' - k(\psi) + \psi + k(\psi)] + k'(\psi)\psi' - \psi' = -a\tau\psi' - a\psi + k'(\psi)\psi' - \psi'$$

and from this last expression a more familiar looking ODE of Liénard type is found:

$$\tau\psi'' + [a\tau + 1 - k'(\psi)]\psi' + a\psi = 0$$

Now, using the trick $a\tau = a\tau - a\tau^* + a\tau^*$, one has $a\tau + 1 = a\tau - a\tau^* + a\tau^* + 1$, and remembering the definition $\tau^* = \frac{b-1}{a}$ ( or $b = a\tau^* + 1$), one can write $a\tau + 1 = a\tau - a\tau^* + a\tau^* + 1 = a(\tau - \tau^*) + a\tau^* + 1 = a\varepsilon + b$ to yield

$$\tau\psi'' + [a\varepsilon + b - k'(\psi)]\psi' + a\psi = 0$$

A straightforward application of the Olech and Levinson-Smith theorems (see e.g. [4, p. 52 ]) shows that for $\varepsilon > 0$ the equilibrium point is globally asymptotically stable, while for $\varepsilon < 0$ there exists a unique stable limit cycle. In other words, this model exhibits an asymptotic behaviour with either a stable fixed point or a stable limit cycle. Transition from one to another is achieved through a neutral Hopf bifurcation arising when the time lag involved in building expectatives is reduced. In the limit case, as this time lag becomes extremely small, i.e. when chartists revise their estimate of the price trend infinitely rapidly, slow-fast cycles limit appear.

## 3. NUMERICAL SIMULATIONS OF NONLINEAR MODEL

In this section our goal is to select a specific chartist demand function in order to simulate the dynamics of the model.

The qualitative form chosen for the chartist demand function has led to an special choice for the function $h$, defined in the following way:

$$h(s) = \frac{b(e^{2(s+g)} - 1)}{e^{2(s+g)} + 1} - \frac{b(e^{2g} - 1)}{e^{2g} + 1}$$

which for the value $s = \psi - g$ yields the following expressions:

$$\begin{aligned} h(\psi - g) &= \frac{b(e^{2\psi} - 1)}{e^{2\psi} + 1} - \frac{b(e^{2g} - 1)}{e^{2g} + 1} \\ k(\psi) &= \frac{b(e^{2\psi} - 1)}{e^{2\psi} + 1} \end{aligned}$$

It is clear (see Fig 1) that the graph of this function is obtained by modifying the well-known function $Th(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}}$ in order to fulfill the properties requested in the model. Now the second order ODE becomes

$$\tau\psi'' + [a\varepsilon + b - \frac{4be^{2\psi}}{(e^{2\psi} + 1)^2}]\psi' + a\psi = 0$$

and the equivalent system for phase-plane analysis is:

$$\begin{aligned} \psi' &= \phi \\ \phi' &= -\frac{a\psi}{\tau} - [a\varepsilon + b - \frac{4be^{2\psi}}{(e^{2\psi} + 1)^2}]\frac{\phi}{\tau} \end{aligned}$$

The numerical simulations carried out (see Fig 2a, 2b, 2c) coincide in detail with the predictions (Hopf bifurcation) of the qualitative analysis of the model.

As it was pointed out, the appearance of a small parameter in the second equation suggests a slow-fast cyclic behaviour for this model.This can be easily checked by applying a separation condition for the existence of limit cycles in slow-fast systems (see [5]). This separation principle, based on singular perturbation arguments, is purely geometric: If the nullcline $p = \frac{k(\psi)}{a}$ of the slow variable separates both stable branches of the nullcline $p = \frac{k(\psi) - \psi}{a}$ of the fast variable, as sketched in Fig.2, a stable cycle exists in the limit case. Numerical simulations (see Fig. 3a, 3b, 3c, 4a, 4b, 4c) show clearly that this occurs and relaxation oscillations can be easily observed.

The interpretation is the following: Whenever $\tau \to 0$, chartists estimate prices on the basis of only recent past prices, thus generating sudden and violent fluctuations corresponding to the fast part of the cycle. From an economic viewpoint, persistent information gathering will trigger quick reactions from chartist agents.

## 4. THE EFFECT OF NOISE ON SPECULATIVE BEHAVIOUR: SOME CONCLUSIONS

Markets are always under the influence of external fluctuations of a random nature that perturb their dynamics. The observation (see [6], [2]) that the evolution of walrasian prices can be represented via a random walk approach by introducing a Wiener process $W_t = \sqrt{2\sigma}\xi_t$, where $\xi_t$ is a gaussian $\delta$-correlated white noise signal with zero mean and noise intensity $\sigma$, suggests the introduction of random terms in the model, thus becoming a system of stochastic differential equations.

To take into account the effect of this random influence, we replace the parameter $W$ in the deterministic system by a stationary random process $W_t = W + \sqrt{2\sigma}\xi_t$, where $W$ corresponds to the average state of the environment, and $\xi_t$ describes fluctuations of strenght $\sigma$ around it. Thus, by collecting all noise terms on the right hand side, a formal calculation analogous to the one in the deterministic case leads to the following nonlinear stochastic differential equations, where a new parameter, viz. the noise intensity $\sigma$, is added to the Hopf bifurcation parameter $\tau$:

$$\begin{aligned}\psi' &= \phi \\ \tau\phi' &= -a\psi - [a\varepsilon + b - \frac{4be^{2\psi}}{(e^{2\psi}+1)^2}]\phi + \sqrt{2\sigma a^2}\xi_t\end{aligned}$$

Systems of this type have been dealt with by the authors in some other works (see for example [7], [8]) on geomorphological processes, where they showed to be extremely useful and accurate. The aim of the study is to analyze how the qualitative structure is modified under changes in $\sigma$.

A first approach can be obtained by developping the expresion $\frac{4be^{2\psi}}{(e^{2\psi}+1)^2}$ as a McLaurin series and retaining only the first term $b$. This simplifies the system to

$$\begin{aligned}\psi' &= \phi \\ \tau\phi' &= -a\psi - a\varepsilon\phi + \sqrt{2\sigma a^2}\xi_t\end{aligned}$$

where the second equation can be written conveniently in the form

$$\tau\phi' = -(V'(\psi) + a\varepsilon\phi) + \sqrt{2\sigma a^2}\xi_t$$

with $V(\psi) = \frac{a\psi^2}{2}$, a single well potential. Under these assumptions the stationary solution to the Fokker-Planck equation of the system can be obtained in closed form (see [8, p. 156]) whenever $\varepsilon > 0$:

$$p_s(\psi,\phi) = N\exp[-\frac{\varepsilon(a\psi^2+\tau\phi^2)}{2\sigma a}]$$

an elliptic bell shaped density function where $N$ stands for the normalization constant. Sketching several graphs of this stationary probability density (see Fig. 5a, 5b, 5c), we note that the external noise obviously has a disorganizing influence. Indeed, since for $\varepsilon > 0$ in the deterministic case the equilibrium point is globally asymptotically stable, the stationary " probability density " mass will be entirely concentrated on it, e. g., it consists of a delta peak centered on the equilibrium point. Then the effect of noise will flatten and spread this sharp peak, depending on its strength. Nevertheless, as was pointed out in the introduction, the general nonlinear case can not be dealt with so easily in this way, so a numerical attack was chosen.

Numerical simulations show that noise distorts the limit cycle giving rise to a crater-like probability density function and a main feature is that a modulation can be observed in the oscillation frequency of the variables (see Fig. 6a, 6b, 6c, 7a, 7b, 7c). Noise seems to amplify by a factor of two the wavelength of the sample paths: This opens the way to the application of averaging techniques in order to determine when the chartist agents change abruptly their price estimations.

It is to be noted that the effect of noise when $\tau$ is small (e.g. $\tau = 0.02$) is reduced to a minimum, showing that intrinsically irregular behaviours are robust to noise influence.

## References


[1] J. M. Pacheco, Reaction diffusion equations and economic cycles, *Jour. of Math. & Comp. Sci. (Math. Ser.)* **9**(2) 137-144 (1996).

[2] A. Beja and M. B. Goldman, On the dynamic behaviour of prices in disequilibrium, *Journal of Finance,* **XXXV**(2) 235-248 (1980)

[3] C. Chiarella, The dynamics of speculative behaviour, *Annals of Operations Research*, 1992.

[4] H.-W. Lorenz, *Nonlinear dynamical economics and chaotic motion,* Springer-Verlag, Berlin (1983)

[5] S. Muratori and S. Rinaldi, A separation condition for the existence of limit cycles in slow-fast systems, *Appl. Math. Modelling* **15**, 312-318 (1991)

[6] P. Samuelson, Proof that properly anticipated prices fluctuate randomly, *Industrial Management Review* **6** 41-49 (1965)

[7] J. M. Pacheco, A simple modelling of braid-like structure (rill marks) appearing on sandy beaches, *Revista de Geofísica* **48**, 159-164 (1992)

[8] J. M. López, J. M. Pacheco and C. Rodríguez, Un modelo matemático para ciertas estructuras sedimentarias primarias, *Rev. de la Academia Canaria de Ciencias,* **VIII**(1), 9-17 (1996) [Spanish]

[9] C. Gardiner, *Handbook of stochastic methods*, Springer-Verlag, Berlin (1983)



Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira, 35017 Las Palmas, Spain

*E-mail address*: pacheco@dma.ulpgc.es ; jlopez@dma.ulpgc.es ; cesar@dma.ulpgc.es

**FIGURES**

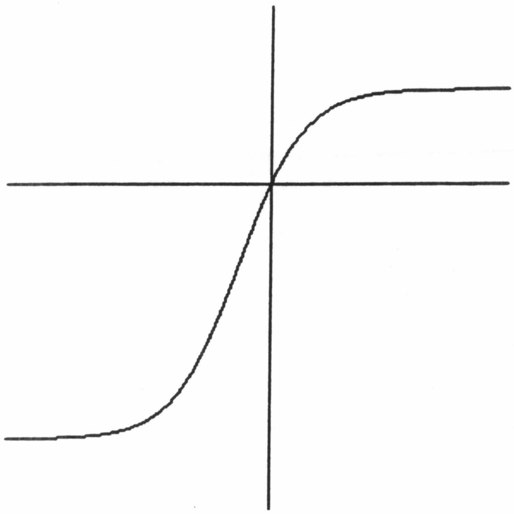


Fig 1 : A typical graph of a chartist demand function

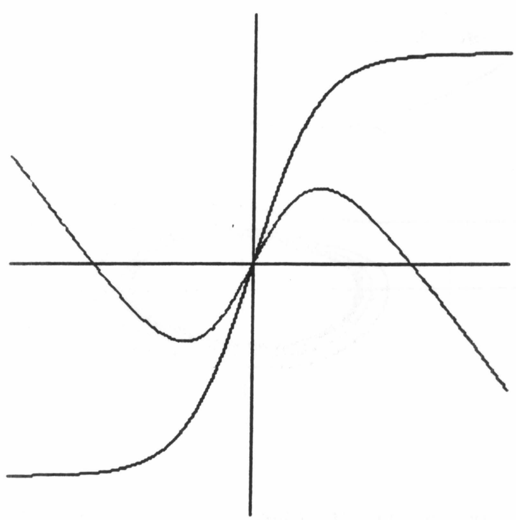


Fig. 3 : Typical isoclines of the dynamics

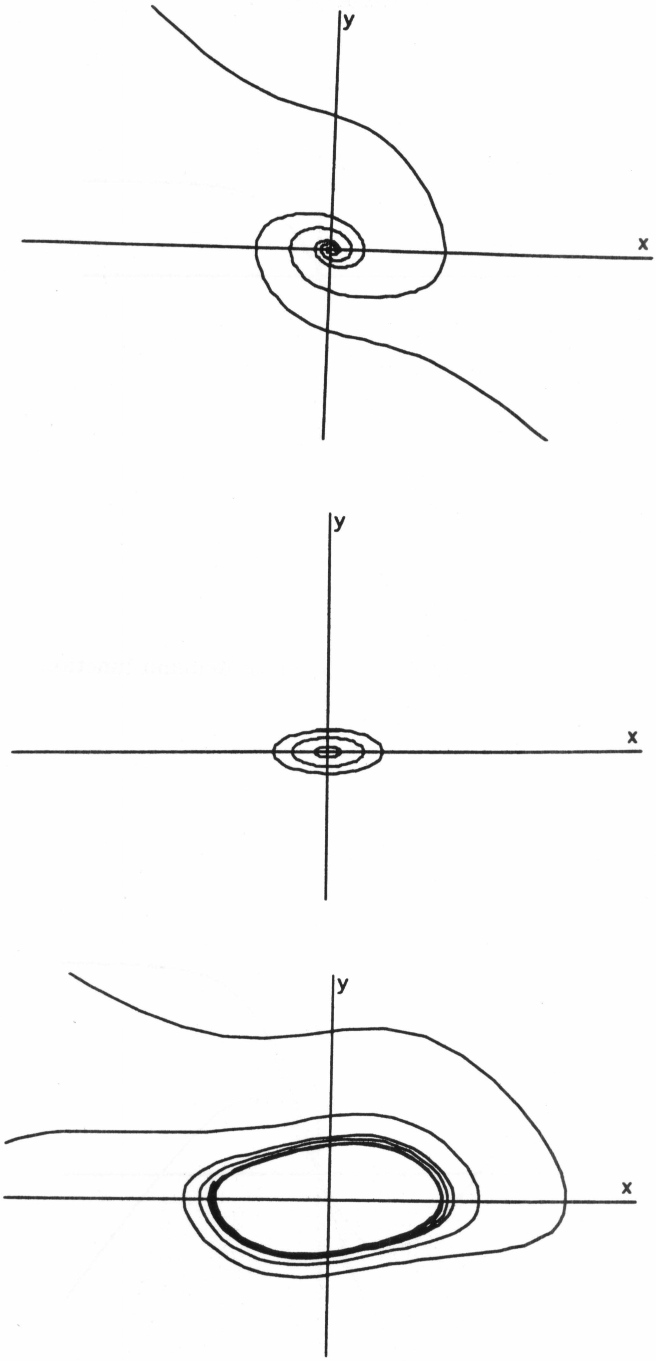
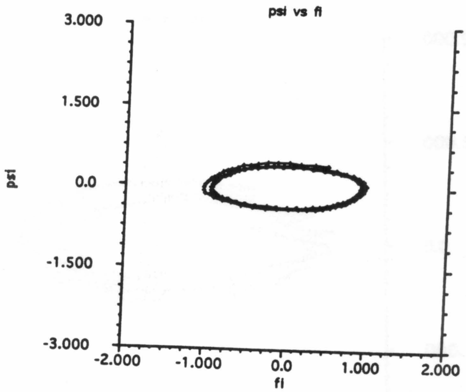
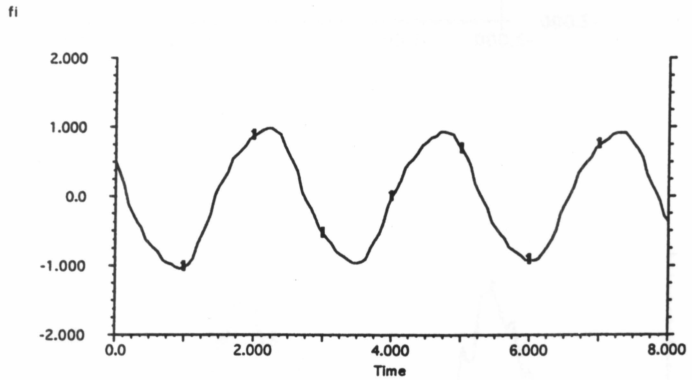



Fig. 2 : The Hopf bifurcation for $a = 0.5$ ; $b = 2$

2a) $\varepsilon = 1$ (spiral sink) ; 2b) $\varepsilon = 0$ (centre) ; 2c) $\varepsilon = -0.5$ (spiral source)




Figs. 4a, 4b : Slow-fast limit cycle for $b = 1.05$, $a = 0.5$ , $\varepsilon = -0.02$ and relaxation oscillations.

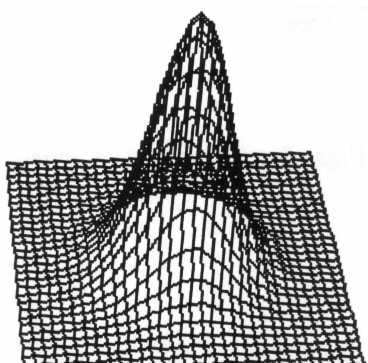
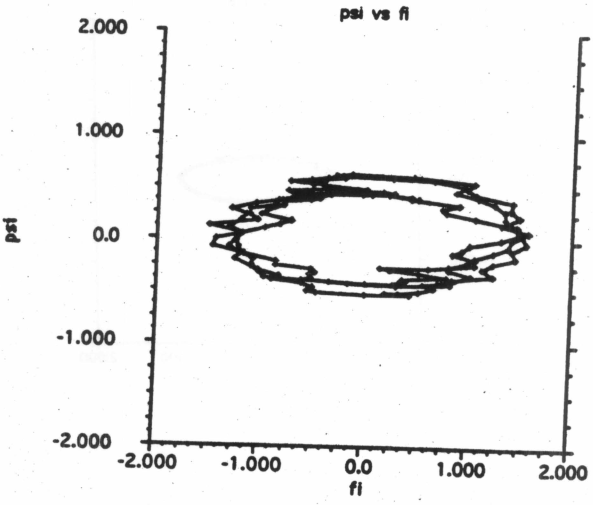
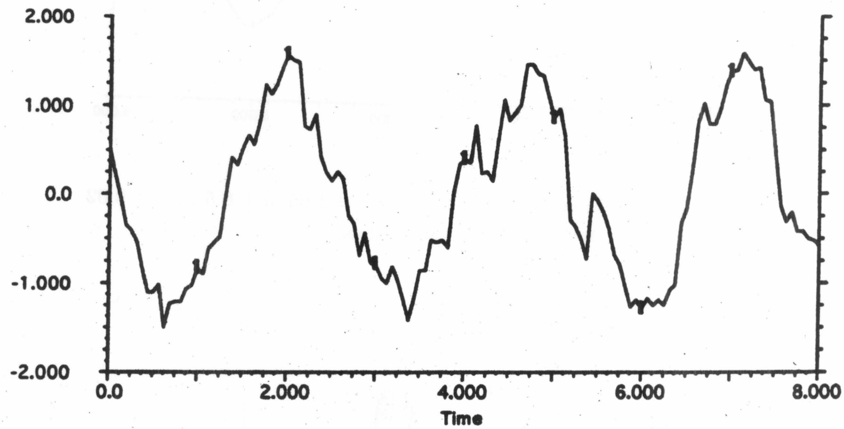
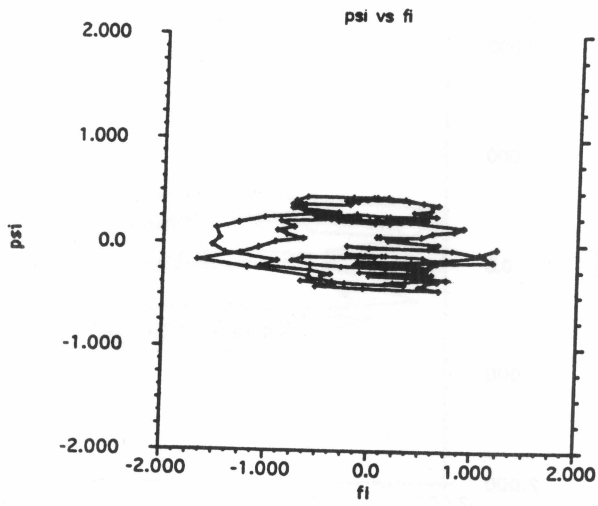


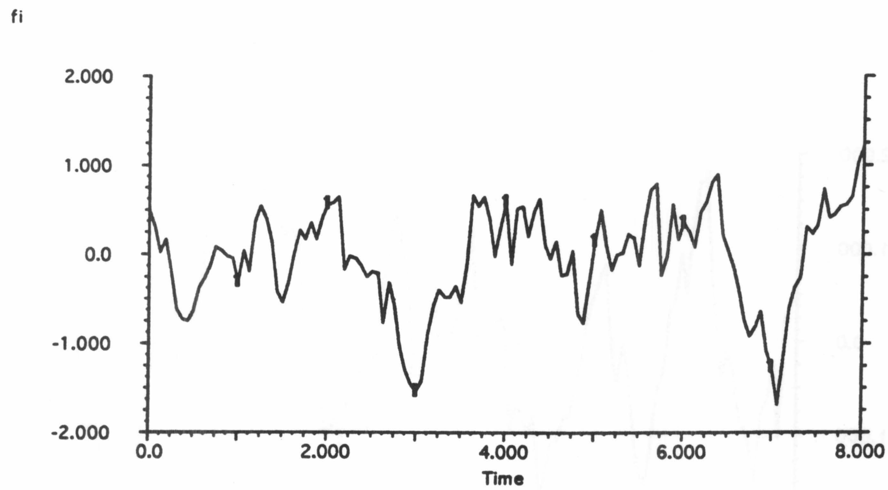
Fig. 5 : A typical stationary probability density for $\varepsilon > 0$.



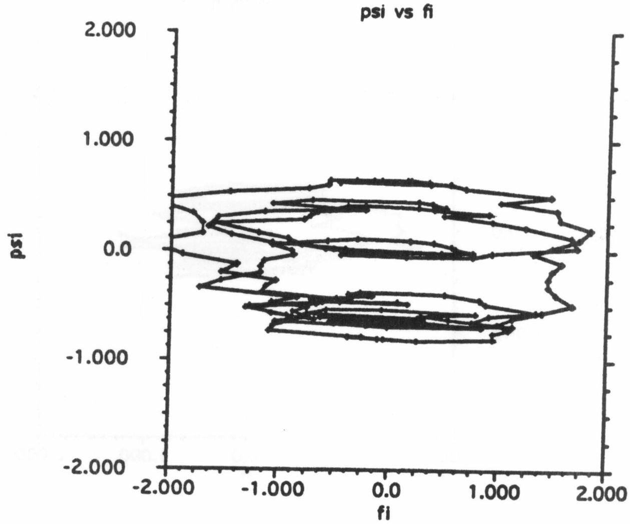



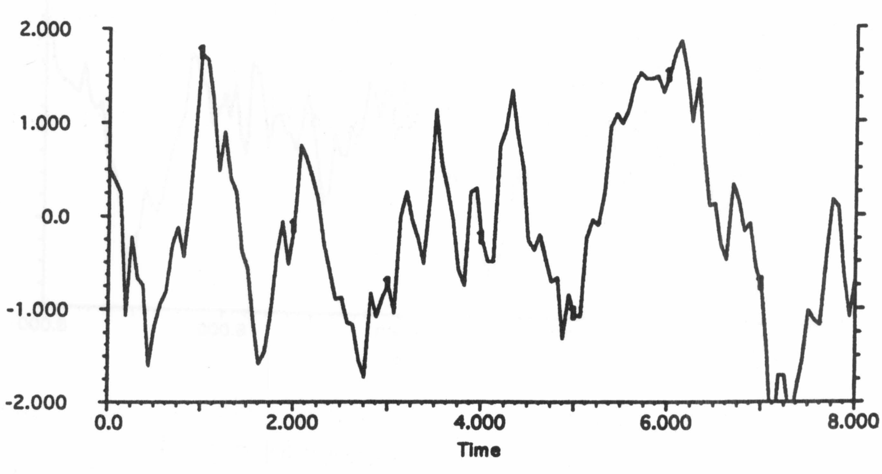

Figs. 6a, 6b : effect of noise for $\sigma = 0.2$




Figs. 7a, 7b : effect of noise for $\sigma = 0.4$




Figs. 8a, 8b : effect of noise for $\sigma = 1$