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INTEGRALS INVOLVING τ-GENERALIZED LEGENDRE FUNCTIONS OF THE FIRST KIND

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ABSTRACT

A fairly wide range of special functions can be represented in terms of the hypergeometric and confluent hypergeometric series. Hypergeometric series in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences. N. Virchenko (1999) has considered the $\tau-$generalization of the Gauss hypergeometric series whereas A. H. Al-Shammery and S. L. Kalla (2001) have extended the idea of $\tau-$generalizations to Appell's hypergeometric functions F_i $(i = 1, 2, 3)$ and they have also considered the $\tau-$generalizations of the confluent hypergeometric functions of two variables Φ_i $(i = 1, 2, 3)$. Recently, L. Galué (2005) has presented one extension of the Humbert functions Ψ_1, Ψ_2, Ξ_1 and Ξ_2 introducing additional parameters τ, τ'. On the other hand, some important computational problems in the field of numerical integration can be solved by procedure which require the modified moments. This paper deals with the modified moments of the weight function $(1-x)^a(1+x)^b(\ln x)^p$, $p = 1, 2$ on $(-1, 1]$ with respect to the product of $\tau-$generalized Legendre functions of the first kind. Various particular cases are also obtained.

Key words: $\tau-$Generalized Legendre functions of the first kind, modified moments, integrals.

RESUMEN

Un gran número de funciones especiales pueden ser representadas en términos de series hipergeométricas y series hipergeométricas confluentes. Las series hipergeométricas en una y más variables aparecen naturalmente en una amplia variedad de problemas en matemática aplicada, estadística, investigación de operaciones, física teórica y ciencias de la ingeniería. N. Virchenko (1999) ha considerado la generalización τ de la serie hipergeométrica de Gauss mientras que A. H. Al-Shammery and S. L. Kalla (2001) han extendido la idea de las generalizaciones τ a las funciones de Appell F_i $(i = 1, 2, 3)$ y además han considerado las generalizaciones τ de las funciones hipergeométricas confluentes de dos variables Φ_i $(i = 1, 2, 3)$. Recientemente, L. Galué (2005) ha presentado una extensión de las funciones de Humbert Ψ_1, Ψ_2, Ξ_1 y Ξ_2 introduciendo parámetros adicionales τ, τ'. Por otro lado, algunos problemas computacionales importantes en el campo de

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integración numérica pueden ser resueltos mediante procedimientos que requieren momentos modificados. Este trabajo trata con los momentos modificados de la función de peso $(1-x)^a(1+x)^b(\ln x)^p$, $p=1,2$ sobre $(-1,1]$ con respecto al producto de funciones de Legendre $\tau-$generalizadas de primera clase. Se obtienen además varios casos particulares.

Palabras Clave: Funciones de Legendre $\tau-$generalizadas de primera clase, momentos modificados, integrales.

1. INTRODUCTION

A fairly wide range of special functions can be represented in terms of the hypergeometric and confluent hypergeometric series. Hypergeometric series in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences ([5],[7],[17],[19],[20],[21]). Exton ([9],[10]) has considered a number of problems, such as, finite and infinite statistical distributions, angular displacement of a shaft, vibration of a thin elastic plate, heat production in a cylinder, dual integral equations, etc., which give rise to integrals associated with hypergeometric series in one and more variables.

In 1999 N. Virchenko [22] has considered the $\tau-$generalization of the Gauss hypergeometric series, whereas A. H. Al-Shammery and S. L. Kalla [3] have extended, in 2001, the idea of $\tau-$generalizations to Appell's hypergeometric functions F_i $(i=1,2,3)$ and they have also considered the $\tau-$generalizations of the confluent hypergeometric functions of two variables Φ_i $(i=1,2,3)$. The $\tau-$confluent hypergeometric function has been used to study a probability distribution ([2], [4]) and inverse Gaussian distribution [1]. Recently, L. Galué (2005) [11] has presented one extension of the Humbert functions Ψ_1, Ψ_2, Ξ_1 and Ξ_2 introducing additional parameters τ, τ'.

On the other hand, some important computational problems in the field of numerical integration can be solved by procedure which require the modified moments [12], [13]. Algorithms concerning the integrals of Legendre polynomials have been presented [6], [14], [16].

Nina A. Virchenko [22] has introduced the $\tau-$generalized Legendre functions of the first kind as:

$$_{\tau}P_k^{m,n}(z) = \frac{1}{\Gamma(1-m)}\frac{(z+1)^{n/2}}{(z-1)^{m/2}} \quad \text{x}$$

$$_2R_1^{\tau}\left(k-\frac{m-n}{2}+1, -k-\frac{m-n}{2}; 1-m; \frac{1-z}{2}\right) \tag{1}$$

where

$$_2R_1^{\tau}(z) = {_2R_1^{\tau}}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!} \tag{2}$$

is a special case of Wright's functions [8], where a,b,c are complex numbers, $\tau \in \mathbb{R}, \tau > 0; a+n \neq 0,-1,-2,...; b+\tau n \neq 0,-1,-2,...,$ when $n=0,1,2,...; a,b,c$ are such that $\Gamma(a+n), \Gamma(b+\tau n), \Gamma(c+\tau n)$ are finite for $n=0,1,2,....$ The serie converges uniformly in $|z|<1$.

From (1) and (2):

$$
{}_\tau P_k^{m,n}(z) = \frac{(z+1)^{n/2}(z-1)^{-m/2}}{\Gamma\left(-k-\frac{m-n}{2}\right)} \quad \text{x}
$$

$$
\sum_{j=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_j \Gamma\left(-k-\frac{m-n}{2}+\tau j\right)}{\Gamma(1-m+\tau j) j!}\left(\frac{1-z}{2}\right)^j, \tag{3}
$$

with

$$
k-\frac{m-n}{2} \neq -1,-2,...;\, -k-\frac{m-n}{2}+\tau j \neq -1,-2,...;
$$

$$
m-\tau j \neq 1,2,...;\; |1-z|<2;\; |\arg(z+1)|<\pi.
$$

The purpose of this paper is evaluate the integral

$$
I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \int_{-1}^{1}(1-x)^a(1+x)^b {}_\tau P_k^{m,n}(x)\; {}_\varepsilon P_h^{M,N}(x)dx \tag{4}
$$

$$
\begin{cases}
\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ \tau,\ \varepsilon\ \in\ \mathbb{R},\ \tau,\ \varepsilon\ >0; \\
-k-\frac{m-n}{2}+\tau j \neq 0,-1,-2,....\ \text{if}\ \tau j \notin \mathbb{N},\ m \notin \mathbb{N}, \\
-h-\frac{M-N}{2}+\varepsilon j \neq 0,-1,-2,....\ \text{if}\ \varepsilon j \notin \mathbb{N},\ M \notin \mathbb{N}, \\
\text{with}\quad j=1,2,3,....,
\end{cases} \tag{5}
$$

and their partial derivatives with respect of a and b, where ${}_\tau P_k^{m,n}(x)$ is defined by

$$
{}_\tau P_k^{m,n}(x) = \frac{1}{\Gamma(1-m)}\frac{(1+x)^{n/2}}{(1-x)^{m/2}} \quad \text{x}
$$

$$
{}_2R_1^\tau\left(k-\frac{m-n}{2}+1,-k-\frac{m-n}{2};1-m;\frac{1-x}{2}\right) \tag{6}
$$

$$
-k-\frac{m-n}{2}+\tau j \neq 0,-1,-2,....\ \text{if}\ \tau j \notin \mathbb{N},\ \text{with}\quad j=1,2,3,....,
$$

$$
m \notin \mathbb{N},\ -1<x<1.
$$

For $\tau = 1$ equation (6) reduces to

$$
P_k^{m,n}(x) = \frac{1}{\Gamma(1-m)}\frac{(1+x)^{n/2}}{(1-x)^{m/2}} \quad \text{x}
$$

$$
{}_2F_1\left(k-\frac{m-n}{2}+1,-k-\frac{m-n}{2};1-m;\frac{1-x}{2}\right), \tag{7}
$$

$$
m \notin \mathbb{N},\ -1<x<1.
$$

If now $n = m$ we get [18, p. 773]

$$
P_k^m(x) = \frac{1}{\Gamma(1-m)}\left(\frac{1+x}{1-x}\right)^{m/2} {}_2F_1\left(k+1,-k;1-m;\frac{1-x}{2}\right), \tag{8}
$$

$$m \notin \mathbb{N}, \ -1 < x < 1.$$

and for $m = 0$ with $k \in \mathbb{N}_0$,

$$P_k^0(x) = P_k(x) = {}_2F_1\left(k+1, -k; 1; \frac{1-x}{2}\right), \ -1 < x < 1. \tag{9}$$

are the Legendre polynomials.

2. EVALUATION OF THE INTEGRALS $\mathbf{I^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}}$

For evaluate (4), we first consider

$$I^{m,n}_{\tau,k,a,b} = \int_{-1}^{1} (1-x)^a (1+x)^b \ {}_\tau P_k^{m,n}(x) dx, \tag{10}$$

where

$$\begin{cases} \mathrm{Re}(a) > 0, \ \mathrm{Re}(b) > 0, \ \tau \in \mathbb{R}, \ \tau > 0, \ m \notin \mathbb{N}, \\ \\ -k - \frac{m-n}{2} + \tau j \neq 0, -1, -2, ... \text{ if } \tau j \notin \mathbb{N}, \text{ with } \ j = 1, 2, 3, ... \end{cases} \tag{11}$$

We substitute ${}_\tau P_k^{m,n}(x)$ from the equations (6) and (2), interchange the order of integral and summation (in view of the absolute convergence) and we have

$$I^{m,n}_{\tau,k,a,b} = \frac{1}{\Gamma\left(-k - \frac{m-n}{2}\right)} \sum_{j=0}^{\infty} \frac{\left(k - \frac{m-n}{2} + 1\right)_j \Gamma\left(-k - \frac{m-n}{2} + \tau j\right)}{\Gamma(1 - m + \tau j) \ j! \ 2^j} \quad \text{x}$$

$$\int_{-1}^{1} (1-x)^{a+j-m/2} \ (1+x)^{b+n/2} \ dx.$$

Using the result [15, p. 285, No. (3.196.3)]

$$\int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} dx = (b-a)^{\mu+\nu-1} B(\mu, \nu)$$

$$b > a, \ \mathrm{Re}(\mu) > 0, \ \mathrm{Re}(\nu) > 0,$$

we get

$$I^{m,n}_{\tau,k,a,b} = \frac{2^{a+b+\frac{n-m}{2}+1} \Gamma\left(b + \frac{n}{2} + 1\right)}{\Gamma\left(-k - \frac{m-n}{2}\right)} \quad \text{x}$$

$$\sum_{j=0}^{\infty} \frac{\left(k - \frac{m-n}{2} + 1\right)_j \Gamma\left(-k - \frac{m-n}{2} + \tau j\right) \Gamma\left(a + j - \frac{m}{2} + 1\right)}{\Gamma(1 - m + \tau j) \ \Gamma\left(a + b - \frac{m-n}{2} + j + 2\right) \ j!}. \tag{12}$$

For evaluate (4) we use a similar procedure to the former.

So, we substitute ${}_{\varepsilon}P_h^{M,N}(x)$ from the equations (6) and (2), then we interchange the order of integral and summation obtaining,

$$I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \frac{1}{\Gamma\left(-h-\frac{M-N}{2}\right)} \sum_{i=0}^{\infty} \frac{\left(h-\frac{M-N}{2}+1\right)_i \, \Gamma\left(-h-\frac{M-N}{2}+\varepsilon i\right)}{\Gamma(1-M+\varepsilon i)\; i!\; 2^i} \quad \text{x}$$

$$\int_{-1}^{1} (1-x)^{a+i-M/2}(1+x)^{b+N/2} \; {}_{\tau}P_k^{m,n}(x)\, dx.$$

From (10) and (12):

$$I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma\left(-k-\frac{m-n}{2}\right)\Gamma\left(-h-\frac{M-N}{2}\right)} \sum_{i,j=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_j}{\Gamma(1-m+\tau j)} \quad \text{x}$$

$$\frac{\Gamma\left(-k-\frac{m-n}{2}+\tau j\right)\Gamma\left(a-\frac{M+m}{2}+i+j+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)} \quad \text{x}$$

$$\frac{\left(h-\frac{M-N}{2}+1\right)_i \, \Gamma\left(-h-\frac{M-N}{2}+\varepsilon i\right)}{\Gamma(1-M+\varepsilon i)\; j!\; i!}, \tag{13}$$

with the conditions given in (5).

If now we derive partially (13) with respect of a, we have

$$J_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \frac{\partial}{\partial a} I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \int_{-1}^{1} (1-x)^a(1+x)^b \ln(1-x) \; {}_{\tau}P_k^{m,n}(x) \; {}_{\varepsilon}P_h^{M,N}(x)dx$$

$$J_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \ln 2 \, I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} + \frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma\left(-k-\frac{m-n}{2}\right)\Gamma\left(-h-\frac{M-N}{2}\right)} \quad \text{x}$$

$$\sum_{i,j=0}^{\infty} \frac{\left(h-\frac{M-N}{2}+1\right)_i}{\Gamma(1-M+\varepsilon i)} \frac{\Gamma\left(-h-\frac{M-N}{2}+\varepsilon i\right)\left(k-\frac{m-n}{2}+1\right)_j}{\Gamma(1-m+\tau j)} \quad \text{x}$$

$$\frac{\Gamma\left(-k-\frac{m-n}{2}+\tau j\right)\Gamma\left(a-\frac{M+m}{2}+i+j+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\; j!\; i!} \left[\psi\left(a-\frac{M+m}{2}+i+j+1\right) - \right.$$

$$\left. \psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right] \tag{14}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the logarithmic derivative of the Gamma function [17].

We also obtain from (13)

$$K_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \frac{\partial}{\partial b} I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \int_{-1}^{1} (1-x)^a(1+x)^b \ln(1+x) \; {}_{\tau}P_k^{m,n}(x) \; {}_{\varepsilon}P_h^{M,N}(x)dx$$

$$K_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} = \ln 2 \, I_{\tau,k,\varepsilon,h,a,b}^{m,n,M,N} + \frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma\left(-k-\frac{m-n}{2}\right)\Gamma\left(-h-\frac{M-N}{2}\right)} \quad \text{x}$$

$$\sum_{i,j=0}^{\infty} \frac{\left(h-\frac{M-N}{2}+1\right)_i}{\Gamma(1-M+\varepsilon i)} \frac{\Gamma\left(-h-\frac{M-N}{2}+\varepsilon i\right)\left(k-\frac{m-n}{2}+1\right)_j}{\Gamma(1-m+\tau j)} \quad \text{x}$$

$$\frac{\Gamma\left(-k-\frac{m-n}{2}+\tau j\right)\Gamma\left(a-\frac{M+m}{2}+i+j+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\, j!\, i!}\left[\psi\left(b+\frac{N+n}{2}+1\right)-\right.$$

$$\left.\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]. \tag{15}$$

From (14):

$$R^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\frac{\partial^2}{\partial a^2}I^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b\ln^2(1-x)\ {}_{\tau}P^{m,n}_{k}(x)\ {}_{\varepsilon}P^{M,N}_{h}(x)dx$$

$$R^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\ln^2 2\, I^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}+\frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma\left(-k-\frac{m-n}{2}\right)\Gamma\left(-h-\frac{M-N}{2}\right)}\quad \text{x}$$

$$\sum_{i,j=0}^{\infty}\frac{\left(h-\frac{M-N}{2}+1\right)_i}{\Gamma(1-M+\varepsilon i)}\frac{\Gamma\left(-h-\frac{M-N}{2}+\varepsilon i\right)\left(k-\frac{m-n}{2}+1\right)_j}{\Gamma(1-m+\tau j)}\quad \text{x}$$

$$\frac{\Gamma\left(-k-\frac{m-n}{2}+\tau j\right)\Gamma\left(a-\frac{M+m}{2}+i+j+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\, j!\, i!}\quad \text{x}$$

$$\left\{2\ln 2\left[\psi\left(a-\frac{M+m}{2}+i+j+1\right)-\right.\right.$$

$$\left.\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]+$$

$$\left[\psi\left(a-\frac{M+m}{2}+i+j+1\right)-\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]^2+$$

$$\left.\psi'\left(a-\frac{M+m}{2}+i+j+1\right)-\psi'\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right\}. \tag{16}$$

From (15):

$$Q^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\frac{\partial^2}{\partial b^2}I^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b\ln^2(1+x)\ {}_{\tau}P^{m,n}_{k}(x)\ {}_{\varepsilon}P^{M,N}_{h}(x)dx$$

$$Q^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\ln^2 2\, I^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}+\frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma\left(-k-\frac{m-n}{2}\right)\Gamma\left(-h-\frac{M-N}{2}\right)}\quad \text{x}$$

$$\sum_{i,j=0}^{\infty}\frac{\left(h-\frac{M-N}{2}+1\right)_i}{\Gamma(1-M+\varepsilon i)}\frac{\Gamma\left(-h-\frac{M-N}{2}+\varepsilon i\right)\left(k-\frac{m-n}{2}+1\right)_j}{\Gamma(1-m+\tau j)}\quad \text{x}$$

$$\frac{\Gamma\left(-k-\frac{m-n}{2}+\tau j\right)\Gamma\left(a-\frac{M+m}{2}+i+j+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\, j!\, i!}\quad \text{x}$$

$$\left\{2\ln 2\left[\psi\left(b+\frac{N+n}{2}+1\right)-\right.\right.$$

$$\left.\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]+$$

$$\left[\psi\left(b+\frac{N+n}{2}+1\right)-\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]^2+$$

$$\psi'\left(b+\frac{N+n}{2}+1\right)-\psi'\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\Bigg\}. \quad (17)$$

From (14) and (15) we have

$$S^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b\ln(1-x^2)\ {}_\tau P_k^{m,n}(x)\ {}_\varepsilon P_h^{M,N}(x)dx$$
$$=J^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}+K^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}. \quad (18)$$

$$T^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b\ln\left(\frac{1-x}{1+x}\right)\ {}_\tau P_k^{m,n}(x)\,{}_\varepsilon P_h^{M,N}(x)dx$$
$$=J^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}-K^{m,n,M,N}_{\tau,k,\varepsilon,h,a,b}. \quad (19)$$

The results (14)-(19) are valid under the conditions given in (5).

3. PARTICULAR CASES

a) If $\tau=\varepsilon=1$ in (13)-(15) and in (18)-(19):

$$I^{m,n,M,N}_{1,k,1,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^bP_k^{m,n}(x)P_h^{M,N}(x)dx$$

$$=\frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)\Gamma\left(a-\frac{M+m}{2}+1\right)}{\Gamma(1-M)\Gamma(1-m)\,\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+2\right)}\ \mathrm{x}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{array}{l} a-\frac{M+m}{2}+1:h-\frac{M-N}{2}+1,-h-\frac{M-N}{2};k-\frac{m-n}{2}+1,\ -k-\frac{m-n}{2};\\ a+b-\frac{M+m}{2}+\frac{N+n}{2}+2:1-M;\ 1-m; \end{array} 1,1\right],\ \mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ m,M\notin\mathbb{N} \quad (20)$$

where $F^{p:\ q;\ k}_{l:\ m;\ n}\left[\begin{array}{l}(a_p):(b_q);(c_k);\\ (\alpha_l):(\beta_m);(\gamma_n);\end{array} x,y\right]$ is the Kampé de Fériet series [20, p. 27].

$$J^{m,n,M,N}_{1,k,1,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b\ln(1-x)\ P_k^{m,n}(x)P_h^{M,N}(x)dx$$

$$J^{m,n,M,N}_{1,k,1,h,a,b}=\ln 2\, I^{m,n,M,N}_{1,k,1,h,a,b}+\frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma(1-M)\,\Gamma(1-m)}\ \mathrm{x}$$

$$\frac{\Gamma\left(a-\frac{M+m}{2}+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+2\right)}\sum_{i,j=0}^{\infty}\frac{\left(h-\frac{M-N}{2}+1\right)_i\left(-h-\frac{M-N}{2}\right)_i}{(1-M)_i\,(1-m)_j}\ \text{x}$$

$$\frac{\left(-k-\frac{m-n}{2}\right)_j\left(k-\frac{m-n}{2}+1\right)_j\left(a-\frac{M+m}{2}+1\right)_{i+j}}{\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+2\right)_{i+j}\,j!\,i!}\left[\psi\left(a-\frac{M+m}{2}+i+j+1\right)\right.$$

$$\left.-\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]. \tag{21}$$

$$K^{m,n,M,N}_{1,k,1,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b\ln(1+x)\;P^{m,n}_k(x)\;P^{M,N}_h(x)dx$$

$$K^{m,n,M,N}_{1,k,1,h,a,b}=\ln 2\,I^{m,n,M,N}_{1,k,1,h,a,b}+\frac{2^{a+b-\frac{M+m}{2}+\frac{N+n}{2}+1}\Gamma\left(b+\frac{N+n}{2}+1\right)}{\Gamma(1-M)\,\Gamma(1-m)}\ \text{x}$$

$$\frac{\Gamma\left(a-\frac{M+m}{2}+1\right)}{\Gamma\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+2\right)}\sum_{i,j=0}^{\infty}\frac{\left(h-\frac{M-N}{2}+1\right)_i\left(-h-\frac{M-N}{2}\right)_i}{(1-M)_i\,(1-m)_j}\ \text{x}$$

$$\frac{\left(k-\frac{m-n}{2}+1\right)_j\left(-k-\frac{m-n}{2}\right)_j\left(a-\frac{M+m}{2}+1\right)_{i+j}}{\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+2\right)_{i+j}\,j!\,i!}\left[\psi\left(b+\frac{N+n}{2}+1\right)-\right.$$

$$\left.\psi\left(a+b-\frac{M+m}{2}+\frac{N+n}{2}+i+j+2\right)\right]. \tag{22}$$

$$\begin{aligned}S^{m,n,M,N}_{1,k,1,h,a,b}&=\int_{-1}^{1}(1-x)^a(1+x)^b\ln(1-x^2)\;P^{m,n}_k(x)\;P^{M,N}_h(x)dx\\&=J^{m,n,M,N}_{1,k,1,h,a,b}+K^{m,n,M,N}_{1,k,1,h,a,b}.\end{aligned} \tag{23}$$

$$\begin{aligned}T^{m,n,M,N}_{1,k,1,h,a,b}&=\int_{-1}^{1}(1-x)^a(1+x)^b\ln\left(\frac{1-x}{1+x}\right)\;P^{m,n}_k(x)\,P^{M,N}_h(x)dx\\&=J^{m,n,M,N}_{1,k,1,h,a,b}-K^{m,n,M,N}_{1,k,1,h,a,b}.\end{aligned} \tag{24}$$

The results (22)-(25) are valid under the conditions given in (21).

b) If $n=m$ y $N=M$ in (20)-(25):

$$I^{m,m,M,M}_{1,k,1,h,a,b}=\int_{-1}^{1}(1-x)^a(1+x)^b P^m_k(x)P^M_h(x)dx$$

$$=\frac{2^{a+b+1}\Gamma\left(b+\frac{M+m}{2}+1\right)\Gamma\left(a-\frac{M+m}{2}+1\right)}{\Gamma(1-M)\Gamma(1-m)\;\Gamma(a+b+2)}\ \text{x}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{array}{lllll} a-\frac{M+m}{2}+1: & h+1, & -h; & k+1, & -k; \\ & & & & \quad 1,1 \\ a+b+2: & 1-M; & 1-m; & & \end{array}\right] \tag{25}$$

$$\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ m,M\notin\mathbb{N}. \tag{26}$$

$$J^{m,m,M,M}_{1,k,1,h,a,b} = \int_{-1}^{1} (1-x)^a (1+x)^b \ln(1-x)\ P_k^m(x) P_h^M(x) dx$$

$$J^{m,m,M,M}_{1,k,1,h,a,b} = \ln 2\, I^{m,m,M,M}_{1,k,1,h,a,b} + \frac{2^{a+b+1}\Gamma\left(b+\frac{M+m}{2}+1\right)\Gamma\left(a-\frac{M+m}{2}+1\right)}{\Gamma(1-M)\Gamma(1-m)\Gamma(a+b+2)} \text{ x}$$

$$\sum_{i,j=0}^{\infty} \frac{(h+1)_i\ (-h)_i\,(k+1)_j\ (-k)_j\left(a-\frac{M+m}{2}+1\right)_{i+j}}{(1-M)_i\ (1-m)_j\ (a+b+2)_{i+j}\ j!\ i!} \text{ x}$$

$$\left[\psi\left(a-\frac{M+m}{2}+i+j+1\right)-\psi(a+b+i+j+2)\right]. \tag{27}$$

$$K^{m,m,M,M}_{1,k,1,h,a,b} = \int_{-1}^{1} (1-x)^a (1+x)^b \ln(1+x)\ P_k^m(x)\ P_h^M(x) dx$$

$$K^{m,m,M,M}_{1,k,1,h,a,b} = \ln 2\, I^{m,m,M,M}_{1,k,1,h,a,b} + \frac{2^{a+b+1}\Gamma\left(b+\frac{M+m}{2}+1\right)\Gamma\left(a-\frac{M+m}{2}+1\right)}{\Gamma(1-M)\Gamma(1-m)\Gamma(a+b+2)} \text{ x}$$

$$\sum_{i,j=0}^{\infty} \frac{(h+1)_i\ (-h)_i\,(k+1)_j\ (-k)_j\left(a-\frac{M+m}{2}+1\right)_{i+j}}{(1-M)_i\ (1-m)_j\ (a+b+2)_{i+j}\ j!\ i!} \text{ x}$$

$$\left[\psi\left(b+\frac{M+m}{2}+1\right)-\psi(a+b+i+j+2)\right]. \tag{28}$$

$$\begin{aligned} S^{m,m,M,M}_{1,k,1,h,a,b} &= \int_{-1}^{1} (1-x)^a (1+x)^b \ln(1-x^2)\ P_k^m(x)\ P_h^M(x) dx \\ &= J^{m,m,M,M}_{1,k,1,h,a,b} + K^{m,m,M,M}_{1,k,1,h,a,b}. \end{aligned} \tag{29}$$

$$\begin{aligned} T^{m,m,M,M}_{1,k,1,h,a,b} &= \int_{-1}^{1} (1-x)^a (1+x)^b \ln\left(\frac{1-x}{1+x}\right) P_k^m(x)\, P_h^M(x) dx \\ &= J^{m,m,M,M}_{1,k,1,h,a,b} - K^{m,m,M,M}_{1,k,1,h,a,b}. \end{aligned} \tag{30}$$

The results (28)-(31) are valid under the conditions given in (27).
c) If $m = M = 0,\ k = n$ and $h = m$ in (26)-(31) with m and $n \in \mathbb{N}_0$:

$$I^{0,0,0,0}_{1,n,1,m,a,b} = \int_{-1}^{1} (1-x)^a (1+x)^b P_n(x) P_m(x) dx$$

$$= \frac{2^{a+b+1}\Gamma(b+1)\Gamma(a+1)}{\Gamma(a+b+2)} \text{ x}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} a+1: & m+1, & -m; & n+1, & -n; \\ & & & & & 1,1 \\ a+b+2: & 1; & & 1; & \end{matrix}\right] \tag{31}$$

$$\text{Re}(a) > 0,\ \text{Re}(b) > 0,\ m,n \in \mathbb{N}_0. \tag{32}$$

$$J^{0,0,0,0}_{1,n,1,m,a,b} = \int_{-1}^{1} (1-x)^a (1+x)^b \ln(1-x) P_n(x) P_m(x) dx$$

$$J^{0,0,0,0}_{1,n,1,m,a,b} = \ln 2\, I^{0,0,0,0}_{1,n,1,m,a,b} + \frac{2^{a+b+1}\Gamma(b+1)\Gamma(a+1)}{\Gamma(a+b+2)} \text{ x}$$

$$\sum_{i,j=0}^{\infty} \frac{(m+1)_i\ (-m)_i\,(n+1)_j\ (-n)_j\,(a+1)_{i+j}}{(1)_i\ (1)_j\ \ (a+b+2)_{i+j}\ \ j!\ i!} \text{ x}$$

$$[\psi(a+i+j+1) - \psi(a+b+i+j+2)]\,. \tag{33}$$

$$K^{0,0,0,0}_{1,n,1,m,a,b} = \int_{-1}^{1} (1-x)^a(1+x)^b \ln(1+x)\ P_n(x)P_m(x)dx$$

$$K^{0,0,0,0}_{1,n,1,m,a,b} = \ln 2\, I^{0,0,0,0}_{1,n,1,m,a,b} + \frac{2^{a+b+1}\Gamma(b+1)\Gamma(a+1)}{\Gamma(a+b+2)} \text{ x}$$

$$\sum_{i,j=0}^{\infty} \frac{(m+1)_i\ (-m)_i\,(n+1)_j\ (-n)_j\,(a+1)_{i+j}}{(1)_i\ (1)_j\ \ (a+b+2)_{i+j}\ \ j!\ i!} \text{ x}$$

$$[\psi(b+1) - \psi(a+b+i+j+2)]\,. \tag{34}$$

$$\begin{aligned} S^{0,0,0,0}_{1,n,1,m,a,b} &= \int_{-1}^{1} (1-x)^a(1+x)^b \ln(1-x^2)\ P_n(x)P_m(x)dx \\ &= J^{0,0,0,0}_{1,n,1,m,a,b} + K^{0,0,0,0}_{1,n,1,m,a,b}. \end{aligned} \tag{35}$$

$$\begin{aligned} T^{0,0,0,0}_{1,n,1,m,a,b} &= \int_{-1}^{1} (1-x)^a(1+x)^b \ln\left(\frac{1-x}{1+x}\right)\ P_n(x)P_m(x)dx \\ &= J^{0,0,0,0}_{1,n,1,m,a,b} - K^{0,0,0,0}_{1,n,1,m,a,b}. \end{aligned} \tag{36}$$

The results (34)-(37) are valid under the conditions given in (33).

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REFERENCES

[1] **Ali, I., Kalla, S.L. and Khajah, H.G.** A generalized inverse Gaussian distribution with $\tau-$confluent hypergeometric function, Integral Transform and Special Functions, **12** (2) (2001), 101-114.

[2] **Al-Saqabi, B. N., Kalla, S. L. and Shafea, A.** On a probability distribution involving a $\tau-$confluent hypergeometric function of two variables. Algebras Groups and Geometries **19** (2002), 254-276.

[3] **Al-Shammery, A.H. and Kalla, S.L.** An extension of some hypergeometric functions of two variables. Rev. Acad. Canar. Cienc., **XII** (2001), 189-196.

[4] **Al-Zamel, A.** On a generalized gamma type distribution with τ-confluent hypergeometric function, Kuwait J. Sci. Eng., **28** (2001), 25-36.

[5] **Andrews, L.C.** *Special Functions of Mathematics for Engineers*, McGraw-Hill, New York (1992).

[6] **Blue, J. L.** A Legendre polynomial integral. Math. Comp., **33** (1979), 739-741.

[7] **Carlson, B.C.** *Special Functions of Applied Mathematics.* Academic Press, New York (1977).

[8] **Dotsenko, M.** On some applications of Wright's hypergeometric functions. C.R. Acad. Bulgare Sci. **44** (1991), 13-16.

[9] **Exton, H.** *Multiple Hypergeometric Functions and Applications*, John Wiley & Sons, New York (1976).

[10] **Exton, H.** *Handbook of Hypergeometric Integrals. Theory, Applications, Tables, Computer Programs,* John Wiley & Sons, New York (1978).

[11] **Galué, L.** An extension of some Humbert′s functions. International Journal of Applied Mathematics, **17** (2005), 91-106.

[12] **Gautschi, W.** On the construction of Gaussian quadrature rules from modified moments. Math. Comp., **24** (1970), 245-260.

[13] **Gautschi, W.** Questions of numerical condition related to polynomials, in Symposium on Recent Advances in Numerical Analysis. Academic Press, New York, 1978, 45-72.

[14] **Gautschi, W.** On the Preceding Paper "A Legendre polynomial integral" by James L. Blue, Math. Comp., **33** (1979), 742-743.

[15] **Gradshteyn, I.S. and Ryzhik, I.M.** Tables of Integrals, Series and Products. Academic Press, New York, 1980.

[16] **Kalla, S.L. and Conde S.** On a Legendre polynomial integral.Tamkang J. Math., **13** (1982), 49-52.

[17] **Lebedev, N.N.** *Special Functions and Their Applications.* Dover Publications Inc., New York, 1972.

[18] **Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I.** Integrals and Series, Vol. 3: More Special Functions. Gordon and Breach Science Publishers, New York, USA, 1990.

[19] **Seaborn, J.B.** *Hypergeometric Functions and Their Applications*, Springer-Verlag, Berlin (1991).
[20] **Srivastava, H.M. and Karlsson, P.W.** *Multiple Gaussian Hypergeometric Series*, John Wiley & Sons, New York (1985).
[21] **Srivastava, H.M. and Kashyap, B.** *Special Functions in Queuing Theory and Related Stochastic Processes*, Academic Press, New York (1982).
[22] **Virchenko, N.A.** On some generalizations of the functions of hypergeometric type. Fractional Calculus & Applied Analysis. Vol. 2, No. 3 (1999), 233-244.