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GROUP-THEORETIC METHOD OF OBTAINING A CLASS OF MIXED TRILATERAL GENERATING RELATIONS FOR CERTAIN SPECIAL FUNCTIONS

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Abstract : In this paper, the authors have made an attempt to present a group-theoretic method of obtaining a class of mixed trilateral generating relations from a given class of bilateral generating relations involving some special functions. Moreover, in course of application of our theorem, we have obtained a good number of results (new and known) on mixed trilateral generating relations for various special functions.

1. INTRODUCTION

Theories in connection with the unification of generating relations (both bilateral and trilateral) of various special functions are of greater importance in the study of special functions. For previous works in this direction, one can see the works [1-7] and [8-11] in connection with the unification of bilateral and trilateral generating relations.

In the present article, the authors have made an attempt to discuss a group-theoretic method on the unification of a class of mixed trilateral generating relations for some special functions, of course when suitable continuous transformation groups can be constructed for those special functions. Furthermore, we would like to point it out that while applying our result on various special functions, we get a good number of theorems on mixed trilateral generating relations for the special functions under consideration.

The detailed discussion is given below :

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2. GROUP-THEORETIC DISCUSSION

Let us first consider a bilateral generating relation involving a particular special function - $p_n^{(\alpha)}(x)$ of degree n and of parameter α as follows :

$$(2.1) \qquad G(x,u,w) = \sum_{n=0}^{\infty} a_n \; p_n^{(\alpha)}(x) \; g_n(u) \; w^n ,$$

where $g_n(u)$ is an arbitrary polynomial of degree n and a_n is independent of x, u, w .

Replacing w by vwz and then multiplying both sides of (2.1) by y^α we get,

$$(2.2) \qquad y^\alpha G(x,u,vwz) = \sum_{n=0}^{\infty} a_n \; (p_n^{(\alpha)}(x) y^\alpha z^n) \; g_n(u) \; (vw)^n .$$

Let us suppose that for the above special function, it is possible to define a linear partial differential operator R, which generates a continuous transformation groups as follows :

$$R = \xi(x,y,z)\frac{\partial}{\partial x} + \eta(x,y,z)\frac{\partial}{\partial y} + \zeta(x,y,z)\frac{\partial}{\partial z} + \theta(x,y,z)$$

such that

$$(2.3) \qquad R(p_n^{(\alpha)}(x) y^\alpha z^n) = \rho_n p_{n+1}^{(\alpha-1)}(x) y^{\alpha-1} z^{n+1}$$

and

$$(2.4) \qquad e^{wR} f(x,y,z) = \Omega(x,y,z) f(g(x,y,z), h(x,y,z), k(x,y,z))$$

Operating both sides of (2.2) by e^{wR} , we get

$$(2.5) \qquad e^{wR}(y^{\alpha}G(x,u,vwz)) = e^{wR}\left(\sum_{n=0}^{\infty} a_n \ (p_n^{(\alpha)}(x)y^{\alpha}z^n) \ g_n(u) \ (vw)^n\right).$$

The left member of (2.5), with the help of (2.4), becomes

$$(2.6) \qquad \Omega(x,y,z)(h(x,y,z))^{\alpha}G(g(x,y,z),u,vwk(x,y,z)).$$

The right member of (2.5), with the help of (2.3), becomes

$$(2.7) \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n \ \frac{w^k}{k!}\rho_n\rho_{n+1}\cdots\rho_{n+k-1}p_{n+k}^{(\alpha-k)}(x) \ y^{\alpha-k}z^{n+k}g_n(u)(vw)^n.$$

Now equating (2.6) and (2.7) and then putting $y = z = 1$, we get

$$(2.8) \qquad \Omega(x,1,1)(h(x,1,1))^{\alpha}G(g(x,1,1),u,vwk(x,1,1))$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n \ \frac{w^{n+k}}{k!}\rho_n\rho_{n+1}\cdots\rho_{n+k-1}p_{n+k}^{(\alpha-k)}(x) \ g_n(u)v^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k} \ w^n \frac{\rho_{n-k}\rho_{n-k+1}\cdots\rho_{n-1}}{k!}p_n^{(\alpha-k)}(x) \ g_{n-k}(u)v^{n-k}$$

$$= \sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} a_{n-k} \ \frac{\rho_{n-k}\cdots\rho_{n-1}}{k!}p_n^{(\alpha-k)}(x) \ g_{n-k}(u)v^{n-k}$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v) \ ,$$

where

$$(2.9) \qquad \sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{\rho_k \rho_{k+1} \cdots \rho_{n-1}}{(n-k)!} p_n^{(\alpha-n+k)}(x) \; g_k(u) v^k$$

Thus we arrive at the following theorem.

Theorem : If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; p_n^{(\alpha)}(x) \; g_n(u) \; w^n$$

then

$$\Omega(x,1,1)(h(x,1,1))^{\alpha} G(g(x,1,1),u,vwk(x,1,1))$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v)$$

where

$$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{\rho_k \rho_{k+1} \cdots \rho_{n-1}}{(n-k)!} p_n^{(\alpha-n+k)}(x) \; g_k(u) v^k$$

3. APPLICATION

Below we give some applications of our result stated in Theorem-1.

(1) At first we take

$$p_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)$$

Then from [12], we see that

$$R = xy^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - xy^{-1}z$$

such that

(3.1) $$R(L_n^{(\alpha)}(x)y^{\alpha}z^{n}) = (n+1)L_{n+1}^{(\alpha-1)}(x)y^{\alpha-1}z^{n+1}$$

and

(3.2) $$e^{wR}f(x,y,z) = \exp(-wxy^{-1}z)f(x(1+wy^{-1}z), y(1+wy^{-1}z), z).$$

Comparing (2.3), (2.4) with (3.1), (3.2), we get

$$\rho_n = (n+1), \qquad \Omega(x,y,z) = \exp(-wxy^{-1}z),$$

$$g(x,y,z) = x(1+wy^{-1}z), \quad h(x,y,z) = y(1+wy^{-1}z), \quad k(x,y,z) = z \quad .$$

Then by the application of our theorem, we get the following result on mixed trilateral generating relation involving Laguerre polynomials.

Corollary 1 : If

(3.3) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; L_n^{(\alpha)}(x) \; g_n(u) \; w^n$$

then

$$(3.4) \qquad \exp(-wx)(1+w)^{\alpha} G(x(1+w), u, vw)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v)$$

where

$$(3.5) \qquad \sigma_n(x,u,v) = \sum_{k=0}^{n} a_n \binom{n}{k} L_n^{(\alpha-n+k)}(x) \; g_k(u) v^k \quad ,$$

which is found derived in [13].

(2) We now take

$$p_n^{(\alpha)}(x) = f_n^{\beta}(x) \quad \text{with} \quad \alpha = \beta .$$

Then from [14], we see that

$$R = xy^{-1} z \frac{\partial}{\partial x} - z \frac{\partial}{\partial y} - (x-1) y^{-1} z$$

such that

$$(3.6) \qquad R(f_n^{\beta}(x) y^{\beta} z^n) = -(n+1) f_{n+1}^{\beta-1}(x) y^{\beta-1} z^{n+1}$$

and

$$(3.7) \qquad e^{wR} f(x,y,z) = \left(\frac{y}{y-zw}\right) \exp\left(\frac{-xzw}{y-zw}\right) f\left(\frac{xy}{y-zw}, y-zw, z\right) .$$

Comparing (2.3), (2.4) with (3.6), (3.7), we get

$$\rho_n = -(n+1), \qquad \Omega(x,y,z) = \left(\frac{y}{y-zw}\right)\exp\left(\frac{-xzw}{y-zw}\right),$$

$$g(x,y,z) = \frac{xy}{y-zw}, \quad h(x,y,z) = y - zw, \quad k(x,y,z) = z \quad .$$

Then by the application of our theorem, we get the following result on mixed trilateral generating relation involving modified Laguerre polynomials.

Corollary 2 : If

(3.8) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; f_n^{\beta}(x) \; g_n(u) \; w^n$$

then

(3.9) $$(1+w)^{-1+\beta} \exp\left(\frac{xw}{1+w}\right) G\left(\frac{x}{1+w}, u, vw\right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v)$$

where

(3.10) $$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \; \binom{n}{k} f_n^{(\beta-n+k)}(x) \; g_k(u) v^k \quad ,$$

which does not seem to appear before.

(3) We now take

$$p_n^{(\alpha)}(x) = Y_n^{(\alpha)}(x)$$

Then from [15], we see that

$$R = x^2 y^{-1} z \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} + xy^{-1} z^2 \frac{\partial}{\partial z} + (\beta - x) y^{-1} z$$

such that

(3.11) $$R(Y_n^{(\alpha)}(x) y^{\alpha} z^n) = \beta Y_{n+1}^{(\alpha-1)}(x) y^{\alpha-1} z^{n+1}$$

and

(3.12) $$e^{wR} f(x,y,z) = (1 - wxy^{-1}z) \exp(\beta w y^{-1} z)$$

$$\times f\left(\frac{x}{1 - wxy^{-1}z}, \frac{y}{1 - wxy^{-1}z}, \frac{z}{1 - wxy^{-1}z}\right).$$

Comparing (2.3), (2.4) with (3.11), (3.12), we get

$$\rho_n = \beta, \qquad \Omega(x,y,z) = (1 - wxy^{-1}z) \exp(\beta w y^{-1} z),$$

$$g(x,y,z) = \frac{x}{1 - wxy^{-1}z}, \quad h(x,y,z) = \frac{y}{1 - wxy^{-1}z}, \quad k(x,y,z) = \frac{z}{1 - wxy^{-1}z}$$

Then by the application of our theorem, we get the following result on mixed trilateral generating relation involving generalised Bessel polynomials.

Corollary 3 : If

$$(3.13) \qquad G(x,u,w)=\sum_{n=0}^{\infty} a_n \; Y_n^{(\alpha)}(x) \; g_n(u) \; w^n$$

then

$$(3.14) \qquad (1-wx)^{1-\alpha} \exp(\beta w) G\left(\frac{x}{1-wx}, u, \frac{vw}{1-wx}\right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v)$$

where

$$(3.15) \qquad \sigma_n(x,u,v)=\sum_{k=0}^{n} a_k \; \frac{\beta^{n-k}}{(n-k)!} Y_n^{(\alpha-n+k)}(x) \; g_k(u) v^k \quad ,$$

which is found derived in [16].

(4) We now take

$$p_n^{(\alpha)}(x)=C_n^{\lambda}(x) \text{ with } \alpha=\lambda$$

Then from [17], we see that

$$R=(x^2-1)y^{-1}z\frac{\partial}{\partial x}+2xz\frac{\partial}{\partial y}-xy^{-1}z$$

such that

$$(3.16) \qquad R(C_n^{\lambda}(x)y^{\lambda}z^n)=\frac{(n+2\lambda-1)(n+1)}{2(\lambda-1)}C_{n+1}^{\lambda-1}(x)y^{\lambda-1}z^{n+1}$$

and

(3.17) $$e^{wR} f(x,y,z) = \{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\}^{-½}$$

$$\times f(x+w(x^2-1)y^{-1}z, y\{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\}, z).$$

Comparing (2.3), (2.4) with (3.16), (3.17), we get

$$\rho_n = \frac{(n+2\lambda-1)(n+1)}{2(\lambda-1)}, \qquad \Omega(x,y,z) = \{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\}^{-½},$$

$$g(x,y,z) = x+w(x^2-1)y^{-1}z, \;\; h(x,y,z) = y\{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\},$$

$$k(x,y,z) = z \quad .$$

Then by the application of our theorem, we get on simplification the following result on mixed trilateral generating relation involving Gegenbauer polynomials.

Corollary 4 : If

(3.18) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; C_n^{\lambda}(x) \; g_n(u) \; w^n$$

then

(3.19) $$\{1+4xw+4(x^2-1)w^2\}^{-½+\lambda} G(x+2w(x^2-1), u, vw)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v)$$

where

(3.20) $$\sigma_n(x,u,v)=\sum_{k=0}^{n} a_k \binom{n}{k}\frac{(-2\lambda-k+1)_{n-k}}{(-\lambda+1)_{n-k}}C_n^{\lambda-n+k}(x)\ g_k(u)v^k \quad ,$$

(5) We now take

$$p_n^{(\alpha)}(x)=P_n^{(\alpha,\beta)}(x)$$

Then from [18], we see that

$$R=(1-x^2)y^{-1}z\frac{\partial}{\partial x}+(1-x)z\frac{\partial}{\partial y}-(1+x)y^{-1}z^2\frac{\partial}{\partial z}-(1+\alpha)(1+x)y^{-1}z$$

such that

(3.21) $$R(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)=-2(n+1)P_{n+1}^{(\alpha,\beta-1)}(x)y^{\beta-1}z^{n+1}$$

and

(3.22) $$e^{wR}f(x,y,z)=\{1+w(1+x)y^{-1}z\}^{-1-\alpha}$$

$$\times f\left(\frac{x+w(1+x)y^{-1}z}{1+w(1+x)y^{-1}z},\frac{y(1+2wy^{-1}z)}{1+w(1+x)y^{-1}z},\frac{z}{1+w(1+x)y^{-1}z}\right).$$

Comparing (2.3), (2.4) with (3.21), (3.22), we get

$$\rho_n=-2(n+1),\qquad \Omega(x,y,z)=\{1+w(1+x)y^{-1}z\}^{-1-\alpha},$$

$$g(x,y,z)=\frac{x+w(1+x)y^{-1}z}{1+w(1+x)y^{-1}z},\quad h(x,y,z)=\frac{y(1+2wy^{-1}z)}{1+w(1+x)y^{-1}z},$$

$$k(x,y,z)=\frac{z}{1+w(1+x)y^{-1}z}$$

Then by the application of our theorem, we get on simplification the following result on mixed trilateral generating relation involving Jacobi polynomials.

Corollary 5 : If

(3.23) $$G(x,u,w)=\sum_{n=0}^{\infty} a_n \; P_n^{(\alpha,\beta)}(x) \; g_n(u) \; w^n$$

then

(3.24) $$\{1-(1+x)\frac{t}{2}\}^{-1-\alpha-\beta}(1-t)^{\beta} G\left(\frac{x-(1+x)\frac{t}{2}}{1-(1+x)\frac{t}{2}}, u, \frac{tv}{1-(1+x)\frac{t}{2}}\right)$$

$$=\sum_{n=0}^{\infty} t^n \sigma_n(x,u,v)$$

where

(3.25) $$\sigma_n(x,u,v)=\sum_{k=0}^{n} a_k \; \binom{n}{k} P_n^{(\alpha,\beta-n+k)}(x) \; g_k(u) v^k$$

Now if in place of R, we consider the following operator [19] :

$$R_1 = (1-x^2)y^{-1}z\frac{\partial}{\partial x} - (1-x)z\frac{\partial}{\partial y} + (1-x)y^{-1}z^2\frac{\partial}{\partial z} + (1-x)y^{-1}z(1+\beta)$$

such that

(3.26) $$R_1(P_n^{(\alpha,\beta)}(x)y^\alpha z^n) = -2(n+1)P_{n+1}^{(\alpha-1,\beta)}(x)y^{\alpha-1}z^{n+1}$$

and

(3.27) $$e^{wR_1} f(x,y,z) = \left(\frac{y}{y+w(x-1)z}\right)^{1+\beta}$$

$$\times f\left(\frac{xy-(-1+x)wz}{y+w(x-1)z}, \frac{y(y-2wz)}{y+w(x-1)z}, \frac{yz}{y+w(x-1)z}\right)$$

then we get the following corollary.

Corollary 6 : If

(3.28) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; P_n^{(\alpha,\beta)}(x) \; g_n(u) \; w^n$$

then

(3.29) $$\{1-(x-1)\frac{t}{2}\}^{-1-\alpha-\beta}(1+t)^\alpha G\left(\frac{x+(x-1)\frac{t}{2}}{1-(x-1)\frac{t}{2}}, u, \frac{vt}{1-(x-1)\frac{t}{2}}\right)$$

$$= \sum_{n=0}^{\infty} t^n \sigma_n(x,u,v)$$

where

(3.30) $$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \; \binom{n}{k} P_n^{(\alpha-n+k,\beta)}(x) \; g_k(u) v^k$$

It may be pointed out that the corollary 6 can be directly obtained from corollary 5 by using the symmetry relation [20] :

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) .$$

(6) Finally, we take

$$p_n^{(\alpha)}(x) = {}_2F_1(-n, \beta; \upsilon; x) \quad \text{with} \quad \alpha = \upsilon .$$

Then from [21], we see that

$$R = x(1-x)y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - (x\beta+1)y^{-1}z$$

such that

(3.31) $$R({}_2F_1(-n,\beta;\upsilon;x)y^{\upsilon}z^n) = (\upsilon-1)\,{}_2F_1(-(n+1),\beta;\upsilon-1;x)y^{\upsilon-1}z^{n+1}$$

and

(3.32) $$e^{wR}f(x,y,z) = (1+y^{-1}zw)^{-1}(1+xy^{-1}zw)^{-\beta}$$

$$\times f\left(\frac{x(1+y^{-1}zw)}{1+xy^{-1}zw}, y(1+y^{-1}zw), z\right).$$

Comparing (2.3), (2.4) with (3.31), (3.32), we get

$$\rho_n = (\upsilon - 1), \qquad \Omega(x,y,z) = (1 + y^{-1}zw)^{-1}(1 + xy^{-1}zw)^{-\beta},$$

$$g(x,y,z) = \frac{x(1+y^{-1}zw)}{1+xy^{-1}zw}, \quad h(x,y,z) = y(1+y^{-1}zw),$$

$$k(x,y,z) = z \quad .$$

Then by the application of our theorem, we get the following result on mixed trilateral generating relation involving Hypergeometric polynomials.

Corollary 7 : If

(3.33) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; {}_2F_1(-n,\beta;\upsilon;x) \; g_n(u) \; w^n$$

then

(3.34) $$(1-w)^{\upsilon-1}(1-xw)^{\beta} G\left(\frac{x(1-w)}{1-xw}, u, vw\right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v)$$

where

(3.35) $$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{(-1+\upsilon)_{n-k}}{(n-k)!} {}_2F_1(-n,\beta;\upsilon-n+k;x) \; g_k(u) v^k \quad .$$

which does not seem to have appeared in the earlier works.

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