Rev. Acad. Canar. Cienc., XVII (Núms. 1-2), 37-58 (2005) (publicado en agosto de 2006)

SOME FAMILIES OF MULTILATERAL GENERATING FUNCTIONS FOR THE GEGENBAUER (OR ULTRASPHERICAL) POLYNOMIALS AND ASSOCIATED HYPERGEOMETRIC POLYNOMIALS

Shy-Der Lin
Department of Applied Mathematics
Chung Yuan Christian University
Chung-Li 32023, Taiwan, Republic of China
E-Mail: shyder@cycu.edu.tw

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
E-Mail: harimsri@math.uvic.ca

Pin-Yu Wang
Department of Mechanical Engineering
Nan-Ya Institute of Technology
Chung-Li 32023, Taiwan, Republic of China
E-Mail: pinyu@nanya.edu.tw

Abstract

The present investigation is essentially a sequel to several earlier works which appeared recently in this *Revista de la Academia Canaria de Ciencias.* The authors begin by showing that a family of bilateral generating functions for the Gegenbauer (or ultraspherical) polynomials, which was stated incorrectly and claimed to have been proved by using the familiar group-theoretic (Lie algebraic) method by A. B. Majumdar [*Rev. Acad. Canaria Cienc.* **7** (1995), 111-115], is derivable fairly easily by suitably specializing a known multilateral generating function. A number of other analogous families of bilateral and multilateral generating functions for hypergeometric polynomials, including those considered in many recent papers, are also investigated here.

2000 *Mathematics Subject Classification.* Primary 33C45, 33C20; Secondary 33C65.

Key Words and Phrases. Generating functions, classical orthogonal polynomials, Jacobi polynomials, Hermite polynomials, Gegenbauer (or ultraspherical) polynomials, Bessel polynomials, Laguerre polynomials, Legendre (or spherical) polynomials, Chebyshev polynomials, bilateral and multilateral generating functions, hypergeometric polynomials, Gauss hypergeometric function, Group-theoretic (Lie algebraic) method.

1. Introduction, Definitions and Preliminaries

It is well known that the classical Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$ are defined by the following generating function:

$$(1-2xt+t^2)^{-\nu}=\sum_{n=0}^{\infty}C_n^\nu(x)t^n \tag{1.1}$$

or, explicitly, in terms of a hypergeometric polynomial by

$$C_n^\nu(x)=\binom{\nu+n-1}{n}(2x)^n\,{}_2F_1\left[\begin{array}{c}-\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2};\\ \\ 1-\nu-n;\end{array}\ \frac{1}{x^2}\right], \tag{1.2}$$

where ${}_2F_1$ denotes the familiar (Gauss's) hypergeometric function which corresponds to the special case

$$u-1=v=1$$

of the generalized hypergeometric function ${}_uF_v$ with u numerator and v denominator parameters. These polynomials are a generalization of the Legendre (or spherical) polynomials $P_n(x)$ (for $\nu=\frac{1}{2}$) as well as the Chebyshev polynomials $T_n(x)$ (for $\nu\to 0$) and $U_n(x)$ (for $\nu=1$) of the first and second kinds, and are *orthogonal* over the interval $(-1,1)$ with respect to the weight function:

$$w_\nu(x):=\left(1-x^2\right)^{\nu-\frac{1}{2}}\qquad\left(\Re(\nu)>-\frac{1}{2}\right).$$

More generally, in terms of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n in x (and with parameters or indices α and β), which are defined usually by

$$\begin{aligned}P_n^{(\alpha,\beta)}(x)&:=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x+1}{2}\right)^{n-k}\left(\frac{x-1}{2}\right)^k\\&=\binom{n+\alpha}{n}{}_2F_1\left(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2}\right)\end{aligned} \tag{1.3}$$

and are *orthogonal* over the interval $(-1,1)$ with respect to the weight function (*cf.*, *e.g.*, Szegö [25, p. 68, Equation (4.3.3)]):

$$w_{\alpha,\beta}(x):=(1-x)^\alpha(1+x)^\beta\qquad\left(\min\{\Re(\alpha),\Re(\beta)\}>-1\right),$$

that is,

$$\begin{aligned}&\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(x)\,dx\\&\qquad=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}\,\delta_{m,n}\end{aligned} \tag{1.4}$$

$$\left(\min\{\Re(\alpha),\Re(\beta)\}>-1;\ m,n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ \mathbb{N}:=\{1,2,3,\ldots\}\right),$$

$\delta_{m,n}$ being the Kronecker symbol, we have the following *special* relationship for the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$:

$$C_n^{\nu+\frac{1}{2}}(x) = \binom{\nu+n}{n}^{-1}\binom{2\nu+n}{n} P_n^{(\nu,\nu)}(x), \tag{1.5}$$

Many other members of the family of classical orthogonal polynomials, including (for example) the Hermite polynomials $H_n(x)$, the Laguerre polynomials $L_n^{(\alpha)}(x)$, and the Bessel polynomials $y_n(x;\alpha,\beta)$, are *also* special or limit cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In particular, for the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} {}_1F_1(-n;\alpha+1;x), \tag{1.6}$$

it is easily observed that [25, p. 103, Equation (5.3.4)]

$$L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}, \tag{1.7}$$

which can indeed be applied to deduce properties and characteristics of the Laguerre polynomials from those of the Jacobi polynomials.

For the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$, by applying the familiar group-theoretic (Lie algebraic) method, the following family of bilateral generating functions was asserted *erroneously* by Majumdar [12, p. 111].

(?) Theorem 1. *If there exists a unilateral generating relation of the form:*

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, C_n^{\lambda-m}(x) w^n, \tag{1.8}$$

then

$$G\big(x+(x^2-1)wyz, wy\big) = \sum_{n=0}^{\infty} C_n^{\lambda-m-p}(x)\,\sigma_n(z)(wy)^n, \tag{1.9}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} \frac{a_p}{(n-p)!}\, \frac{(p-2\lambda+2m+1)_{n-p}\,(p+1)_{n-p}}{2^{n-p}\,(-\lambda+m+1)_p}\, z^{n-p}. \tag{1.10}$$

Here, and in what follows in our present investigation, $(\lambda)_\nu$ denotes the Pochhammer symbol (or the *shifted factorial,* since $(1)_n = n!$ for $n \in \mathbb{N}_0$) defined (for $\lambda,\nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N}:=\mathbb{N}_0\setminus\{0\};\ \lambda\in\mathbb{C}), \end{cases} \tag{1.11}$$

A *closer* examination of (1.8), (1.9) and (1.10) would immediately reveal the fact that the parameters m and y are redundant (or superfluous), since (*without* any loss of generality) we can *trivially* replace λ by $\lambda+m$ and set $y=1$. More importantly, the parameter p occurring on the right-hand side of the assertion (1.9) is *conspicuously* absent on the left-hand side of (1.9). We begin this paper by presenting a *corrected* and *modified* version of (?) Theorem 1, which is also shown here to be derivable fairly easily from one of several much more general known families of *multilateral* generating functions for the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$. We then propose to (systematically) analyze and investigate many other families of (known or new) families of bilinear, bilateral and mixed multilateral generating functions associated with the following class of hypergeometric polynomials:

$$ {}_2F_1(-n,\alpha;\nu;x) = \sum_{k=0}^{n} \frac{(-n)_k\,(\alpha)_k}{(\nu)_k}\,\frac{x^k}{k!} \tag{1.12}$$

$$\left(n \in \mathbb{N}_0 := \{0,1,2,\ldots\};\ \alpha \in \mathbb{C};\ \nu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,\ldots\}\right),$$

which have also been considered recently, from a markedly different viewpoint, by (for example) Driver and Möller [8], Chen and Srivastava [4], and others (see also the references cited in each of these earlier works).

2. Generating Functions for the Gegenbauer (or Ultraspherical) Polynomials

For the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$ introduced in Section 1, the following generating functions are known fairly well (*cf.*, *e.g.*, McBride [13, p, 56]):

$$\sum_{k=0}^{n} \binom{k-n-2\nu}{k} C_{n-k}^\nu(x)\, t^k = R^n\, C_n^\nu\left(\frac{x-t}{R}\right), \tag{2.1}$$

$$\sum_{k=0}^{\infty} \binom{n+k}{k} C_{n+k}^\nu(x)\, t^k = R^{-n-2\nu}\, C_n^\nu\left(\frac{x-t}{R}\right), \tag{2.2}$$

and

$$\sum_{k=0}^{\infty} \binom{n+k+2\nu-1}{n} C_k^\nu(x)\, t^k = R^{-n-2\nu}\, C_n^\nu\left(\frac{1-xt}{R}\right), \tag{2.3}$$

where, for convenience, R is given by

$$R := \left(1-2xt+t^2\right)^{\frac{1}{2}}. \tag{2.4}$$

As a matter of fact, in view of the hypergeometric representation (*cf.* Rainville [18, p. 280, Equation (20)]):

$$C_n^\nu(x) = \binom{2\nu+n-1}{n} x^n\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2}; \\ \\ \nu+\frac{1}{2}; \end{matrix}\ \frac{x^2-1}{x^2}\right] \tag{2.5}$$

or, equivalently,

$$C_n^\nu(x) = \binom{2\nu+n-1}{n} x^{-2\nu-n} {}_2F_1\left[\begin{array}{l} \nu+\frac{1}{2}n, \quad \nu+\frac{1}{2}n+\frac{1}{2}; \\ \qquad\qquad\qquad\qquad \dfrac{x^2-1}{x^2} \\ \qquad\qquad \nu+\frac{1}{2}; \end{array}\right], \tag{2.6}$$

which follows from (2.6) by means of Euler's transformation:

$$_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,\, c-b;\, c;\, z) \tag{2.7}$$

$$\left(|\arg(1-z)| \leqq \pi - \varepsilon; \quad 0 < \varepsilon < \pi; \quad a,b \in \mathbb{C}; \quad c \in \mathbb{C}\setminus\mathbb{Z}_0^-\right),$$

it is not difficult to show that the generating function (2.3) is actually a *special* case of the following well-known result [23, p. 126, Equation 2.4 (9)]:

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{(2\nu)_k} C_k^\nu(x)\, t^k = (1-xt)^{-\lambda}\, {}_2F_1\left[\begin{array}{l} \frac{1}{2}\lambda, \quad \frac{1}{2}\lambda+\frac{1}{2}; \\ \qquad\qquad\qquad \dfrac{(x^2-1)\,t^2}{(1-xt)^2} \\ \qquad \nu+\frac{1}{2}; \end{array}\right] \tag{2.8}$$

when

$$\lambda = 2\nu + n \quad (n \in \mathbb{N}_0).$$

Next, by applying the relationship (1.5) in its relatively more convenient form:

$$P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(x) = \frac{\left(\nu+\frac{1}{2}\right)_n}{(2\nu)_n} C_n^\nu(x), \tag{2.9}$$

many of the known results involving Jacobi polynomials can easily be specialized to hold true for the Gegenbauer (or ultraspherical) polynomials. Thus, for example, we find from the following known results (see, for example, Srivastava and Manocha [23]; see also González *et al.* [9]):

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-n,\beta-n)}(x) t^n = \left\{1+\frac{1}{2}(x+1)t\right\}^\alpha \left\{1+\frac{1}{2}(x-1)t\right\}^\beta \cdot P_m^{(\alpha,\beta)}\left(x+\frac{1}{2}\left(x^2-1\right)t\right) \tag{2.10}$$

$$\left(m \in \mathbb{N}_0; \quad |t| < \min\left\{2\,|x+1|^{-1}, \ 2\,|x-1|^{-1}\right\}\right),$$

$$\sum_{\ell=0}^{n} \binom{\alpha+\beta+n+\ell}{\ell} P_{n-\ell}^{(\alpha+\ell,\beta+\ell)}(x) t^\ell = P_n^{(\alpha,\beta)}(x+2t), \tag{2.11}$$

$$\sum_{\ell=0}^{\infty} \frac{(\lambda)_\ell}{(-\alpha-\beta)_\ell} P_\ell^{(\alpha-\ell,\beta-\ell)}(x) t^\ell$$

$$= \left\{1+\frac{1}{2}(x-1)t\right\}^{-\lambda} {}_2F_1 \left[\begin{array}{ll} \lambda, \; -\alpha; & \\ & -\dfrac{t}{1+\frac{1}{2}(x-1)t} \\ -\alpha-\beta; & \end{array}\right] \tag{2.12}$$

$$\left(|t| < 2\,|x-1|^{-1}\right),$$

and

$$\sum_{\ell=0}^{n} \frac{(-n)_\ell}{(-\alpha-\beta)_\ell} P_\ell^{(\alpha-\ell,\beta-\ell)}(x) t^\ell$$

$$= \binom{\alpha+\beta}{n}^{-1} t^n P_n^{(\alpha-n,\beta-n)}\left(x+2t^{-1}\right) \tag{2.13}$$

that

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(1-2\nu-m)_n}{(1-\nu)_n} C_{m+n}^{\nu-n}(x)\, t^n$$

$$= \left\{1+4xt+4\left(x^2-1\right) t^2\right\}^{\nu-\frac{1}{2}} C_m^{\nu}\left(x+2\left(x^2-1\right) t\right), \tag{2.14}$$

$$\sum_{\ell=0}^{n} \binom{\nu+\ell-1}{\ell} C_{n-\ell}^{\nu+\ell}(x)\, t^\ell = C_n^{\nu}\left(x+\frac{1}{2}t\right), \tag{2.15}$$

$$\sum_{\ell=0}^{\infty} \frac{(\lambda)_\ell}{(1-\nu)_\ell} C_\ell^{\nu-\ell}(x)\, t^\ell$$

$$= \{1+2(x-1)t\}^{-\lambda} {}_2F_1 \left[\begin{array}{ll} \lambda, \; \frac{1}{2}-\nu; & \\ & -\dfrac{4t}{1+2(x-1)t} \\ 1-2\nu; & \end{array}\right], \tag{2.16}$$

and

$$\sum_{\ell=0}^{n} \binom{n}{\ell} \binom{\ell-\nu}{\ell}^{-1} C_\ell^{\nu-\ell}(x)\, t^\ell = \binom{n-\nu}{n}^{-1} t^n\, C_n^{\nu-n}\left(x - \frac{1}{2t}\right), \tag{2.17}$$

respectively.

The finite summation formulas (2.15) and (2.17) are substantially the same result. Indeed, upon reversing the order of the sum in (2.17), if we replace ν and t by $\nu + n$ and $-t^{-1}$, respectively, we easily obtain the finite summation formula (2.15). More importantly, since [23, p. 126, Equation 2.4 (6)]

$$C_n^\nu(x) = \left(-2\sqrt{x^2-1}\right)^n P_n^{(-\nu-n,\,-\nu-n)}\left(\frac{x}{\sqrt{x^2-1}}\right), \tag{2.18}$$

which, in view of the relationship (2.9), assumes the following *equivalent* form:

$$C_n^\nu(x) = \left(-\frac{1}{2}\sqrt{x^2-1}\right)^n \frac{(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n} C_n^{\frac{1}{2}-\nu-n}\left(\frac{x}{\sqrt{x^2-1}}\right), \tag{2.19}$$

the generating functions (2.2) and (2.14) are equivalent.

Now, in view of the hypergeometric representation (1.2), by appropriately specializing a known hypergeometric generating function [9, p. 138, Equation (2.15)], we can deduce the following generating function (*cf.* [9, p. 163, Equation (6.18)]):

$$\sum_{k=0}^{\infty} \binom{\nu+k-1}{k} C_n^{\nu+k}(x)\, t^k = (1-t)^{-\nu-\frac{1}{2}n}\, C_n^\nu\left(\frac{x}{\sqrt{1-t}}\right). \tag{2.20}$$

Indeed, by means of the relationship (2.19), it is not difficult to rewrite this last generating function (2.20) in its *equivalent* form:

$$\sum_{k=0}^{\infty} \frac{(1-2\nu-n)_{2k}}{(1-\nu)_k} C_n^{\nu-k}(x) \frac{t^k}{k!}$$
$$= (1-4t)^{\nu-\frac{1}{2}} \left\{1 + 4\left(x^2-1\right)t\right\}^{\frac{1}{2}n} C_n^\nu\left(\frac{x}{\sqrt{1+4\left(x^2-1\right)t}}\right). \tag{2.21}$$

Lastly, if we apply the relationship (2.19), we easily see that the finite summation formula (2.15) assumes the equivalent form (2.1). Thus the finite summation formulas (2.1), (2.15), and (2.17) are essentially the same result stated in seemingly different ways.

With a view to obtaining various families of bilinear, bilateral or mixed multilateral generating functions for the Gegenbauer (or ultraspherical) polynomials, we first observe that (2.2) leading to a known result due to Srivastava [20, Part I, p. 229, Corollary 4], as well as each of the generating functions (2.14) [with ν replaced *trivially* by $\nu - m$ $(m \in \mathbb{N}_0)$], (2.20) [with ν replaced *trivially* by $\nu + m$ $(m \in \mathbb{N}_0)$], and (2.21) [with ν replaced *trivially* by $\nu - m$ $(m \in \mathbb{N}_0)$] fits easily into the Singhal-Srivastava definition [19, p. 755, Equation (1)]:

$$\sum_{k=0}^{\infty} A_{m,k}\, S_{m+k}(x)\, t^k = f(x,t)\,\{g(x,t)\}^{-m}\, S_m(h(x,t)) \qquad (m \in \mathbb{N}_0). \tag{2.22}$$

Thus, by comparing the Singhal-Srivastava generating function (2.22) with the aforementioned (trivially modified) versions of the generating functions (2.2), (2.14), (2.20), and (2.21), respectively, we obtain the following special cases of (2.22):

$$A_{m,k} = \binom{m+k}{k}, \quad f = R^{-2\nu}, \quad g = R, \quad h = \frac{x-t}{R}, \quad \text{and} \quad S_k(x) = C_k^{\nu}(x); \tag{2.23}$$

$$\begin{gathered} A_{m,k} = \binom{m+k}{k} \frac{(1-2\nu+m)_k}{(1-\nu+m)_k}, \quad f = \left\{1+4xt+4\left(x^2-1\right)t^2\right\}^{\nu-\frac{1}{2}}, \\ g = 1+4xt+4\left(x^2-1\right)t^2, \quad h = x+2\left(x^2-1\right)t, \\ \text{and} \qquad S_k(x) = C_k^{\nu-k}(x); \end{gathered} \tag{2.24}$$

$$\begin{gathered} A_{m,k} = \binom{\nu+m+k-1}{k}, \quad f = (1-t)^{-\nu-\frac{1}{2}n}, \quad g = 1-t, \\ h = \frac{x}{\sqrt{1-t}}, \quad \text{and} \quad S_k(x) = C_n^{\nu+k}(x); \end{gathered} \tag{2.25}$$

$$\begin{gathered} A_{m,k} = \frac{(1-2\nu+2m-n)_{2k}}{k!(1-\nu+m)_k}, \quad f = (1-4t)^{\nu-\frac{1}{2}}\left\{1+4\left(x^2-1\right)t\right\}^{\frac{1}{2}n}, \\ g = 1-4t, \quad h = \frac{x}{\sqrt{1+4\left(x^2-1\right)t}}, \quad \text{and} \quad S_k(x) = C_n^{\nu-k}(x). \end{gathered} \tag{2.26}$$

In light of the connections exhibited by (2.23) to (2.26), the *entire* development stemming from the Singhal-Srivastava generating function (2.22) would readily apply also to each of the generating functions (2.2), (2.14), (2.20), and (2.21). Alternatively, however, by appealing *directly* to the generating functions (2.2), (2.14), (2.20), and (2.21), we can derive a set of four families of bilinear, bilateral or mixed multilateral generating functions for the Gegenbauer (or ultraspherical) polynomials, which are given by Theorems 3 to 6 below. More importantly, we first make use of the generating function (2.14) in order to derive the following *corrected* and *modified* version of the *erroneous* result asserted by Theorem 1 *without* applying the group-theoretic (Lie algebraic) method.

Theorem 2. *If there exists a unilateral generating relation of the form:*

$$G(x,z) = \sum_{n=0}^{\infty} a_n\, C_n^{\nu-n}(x)\, z^n \qquad (a_n \neq 0), \tag{2.27}$$

then

$$\begin{aligned} \{1+4xt+4(x^2-1)t^2\}^{\nu-\frac{1}{2}}\, G\left(x+(x^2-1)t, \frac{zt}{1+4xt+4(x^2-1)t^2}\right) \\ = \sum_{n=0}^{\infty} C_n^{\nu-n}(x)\, \sigma_n(z)\, t^n, \end{aligned} \tag{2.28}$$

where

$$\sigma_n(z) := \sum_{k=0}^{n} \binom{n}{k} \frac{(1-2\nu+k)_{n-k}}{(1-\nu+k)_{n-k}}\, a_k\, z^k. \tag{2.29}$$

Proof. In order to give a *direct* proof of the assertion (2.28) of Theorem 2 *without* using the group-theoretic (Lie algebraic) method adopted by Majumdar [12] to prove the obviously incorrect Theorem 1, we first substitute the definition (2.29) into the right-hand side of (2.28) and invert the order of the resulting double sum. Then, upon evaluating the inner n-sum by appropriately applying the generating function (2.14), we simply interpret the remaining k-sum by means of (2.27).

Much more general families of bilinear, bilateral or mixed multilateral generating functions than the assertion (2.28) of Theorem 2 are readily accessible in the existing mathematical literature. We first recall a *mild* extension of the aforementioned class of bilateral generating functions of Srivastava [20, Part I, p. 229, Corollary 4] as Theorem 3 below (see also González *et al.* [9]).

Theorem 3. *Corresponding to a non-vanishing function* $\Omega_\mu(y_1, \ldots, y_s)$ *of* s *variables*

$$y_1, \ldots, y_s \qquad (s \in \mathbb{N})$$

and of (complex) order μ, *let*

$$\Lambda^{(1)}_{m,p,q}[x; y_1, \ldots, y_s; z] := \sum_{n=0}^{\infty} a_n\, C^{\nu+\rho q n}_{m+qn}(x)\, \Omega_{\mu+pn}(y_1, \ldots, y_s)\, z^n \tag{2.30}$$

$$(a_n \neq 0; \quad m \in \mathbb{N}_0; \quad p, q \in \mathbb{N}),$$

where ρ *is a suitable complex parameter. Suppose also that*

$$\Upsilon^{\nu,q,\rho}_{n,m,p}(x; y_1, \ldots, y_s; z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k\, C^{\nu+\rho q k}_{m+n}(x) \cdot \Omega_{\mu+pk}(y_1, \ldots, y_s)\, z^k. \tag{2.31}$$

Then

$$\sum_{n=0}^{\infty} \Upsilon^{\nu,q,\rho}_{n,m,p}(x; y_1, \ldots, y_s; z)\, t^n = R^{-m-2\nu}\, \Lambda^{(1)}_{m,p,q}\left[\frac{x-t}{R}; y_1, \ldots, y_s; z\left(\frac{t}{R^{2\rho+1}}\right)^q\right], \tag{2.32}$$

provided that each member of (2.32) *exists,* R *being given, as before, with* (2.4).

We choose also to recall the following analogous families of bilinear, bilateral or mixed multilateral generating functions for the Gegenbauer (or ultraspherical) polynomials (*cf.* González *et al.* [9]).

Theorem 4. *Under the hypotheses of Theorem* 3, *let*

$$\Lambda^{(2)}_{m,p,q}[x; y_1, \ldots, y_s; z] := \sum_{n=0}^{\infty} a_n\, C^{\nu-\rho q n}_{m+qn}(x)\, \Omega_{\mu+pn}(y_1, \ldots, y_s)\, z^n \tag{2.33}$$

$$(a_n \neq 0; \quad m \in \mathbb{N}_0; \quad p, q \in \mathbb{N}),$$

where ρ is a suitable complex parameter. Suppose also that

$$\begin{aligned}
&\mathcal{P}_{n,m,p}^{\nu,q,\rho}(x; y_1, \ldots, y_s; z) \\
&\quad := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} \frac{(1-2\nu-m+(2\rho-1)qk)_{n-qk}}{(1-\nu+\rho qk)_{n-qk}} a_k \, C_{m+n}^{\nu-n-(\rho-1)qk}(x) \\
&\qquad \cdot \Omega_{\mu+pk}(y_1, \ldots, y_s) \, z^k.
\end{aligned} \tag{2.34}$$

Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{P}_{n,m,p}^{\nu,q,\rho}(x; y_1, \ldots, y_s; z) \, t^n \\
&\quad = \left\{1 + 4xt + 4\left(x^2-1\right) t^2\right\}^{\nu-\frac{1}{2}} \\
&\qquad \cdot \Lambda_{m,p,q}^{(2)}\left[x + 2\left(x^2-1\right) t; y_1, \ldots, y_s; \frac{zt^q}{\{1+4xt+4(x^2-1)\,t^2\}^{\rho q}}\right],
\end{aligned} \tag{2.35}$$

provided that each member of (2.35) *exists.*

Theorem 5. *Under the applicable hypotheses of Theorem 3, let*

$$\begin{aligned}
&\Lambda_{n,p,q}^{(3)}[x; y_1, \ldots, y_s; z] \\
&\quad := \sum_{k=0}^{\infty} a_k \, C_n^{\nu+(\rho+1)qk}(x) \, \Omega_{\mu+pk}(y_1, \ldots, y_s) \, z^k
\end{aligned} \tag{2.36}$$

$$(a_k \neq 0; \quad n \in \mathbb{N}_0; \quad p, q \in \mathbb{N}),$$

where ρ is a suitable complex parameter. Suppose also that

$$\begin{aligned}
&\mathcal{Q}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z) \\
&\quad := \sum_{\ell=0}^{[k/q]} \binom{\nu+k+\rho q\ell-1}{k-q\ell} a_\ell \, C_n^{\nu+k+\rho q\ell}(x) \\
&\qquad \cdot \Omega_{\mu+p\ell}(y_1, \ldots, y_s) \, z^\ell.
\end{aligned} \tag{2.37}$$

Then

$$\begin{aligned}
&\sum_{k=0}^{\infty} \mathcal{Q}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z) \, t^k \\
&\quad = (1-t)^{-\nu-\frac{1}{2}n} \, \Lambda_{n,p,q}^{(3)}\left[\frac{x}{\sqrt{1-t}}; y_1, \ldots, y_s; \frac{zt^q}{(1-t)^{(\rho+1)q}}\right],
\end{aligned} \tag{2.38}$$

provided that each member of (2.38) *exists.*

Theorem 6. *Under the applicable hypotheses of Theorem* 3, *let*

$$\Lambda^{(4)}_{n,p,q}[x;\,y_1,\ldots,y_s;\,z] := \sum_{k=0}^{\infty} a_k\, C_n^{\nu-\rho qk}(x)\,\Omega_{\mu+pk}(y_1,\ldots,y_s)\,\frac{z^k}{(qk)!} \tag{2.39}$$

$$(a_k \neq 0;\quad n \in \mathbb{N}_0;\quad p,q \in \mathbb{N}),$$

where ρ *is a suitable complex parameter. Suppose also that*

$$\mathcal{R}^{\nu,\mu,\rho}_{k,p,q}(x;\,y_1,\ldots,y_s;\,z) := \sum_{\ell=0}^{[k/q]} \binom{k}{q\ell} \frac{(1-2\nu+2\rho q\ell-n)_{2(k-q\ell)}}{(1-\nu+\rho q\ell)_{k-q\ell}}\, a_\ell\, C_n^{\nu-k-(\rho-1)q\ell}(x) \cdot \Omega_{\mu+p\ell}(y_1,\ldots,y_s)\, z^\ell. \tag{2.40}$$

Then

$$\sum_{k=0}^{\infty} \mathcal{R}^{\nu,\mu,\rho}_{k,p,q}(x;\,y_1,\ldots,y_s;\,z)\,\frac{t^k}{k!} = (1-4t)^{\nu-\frac{1}{2}}\left\{1+4\left(x^2-1\right)t\right\}^{\frac{1}{2}n} \cdot \Lambda^{(4)}_{n,p,q}\left[\frac{x}{\sqrt{1+4\left(x^2-1\right)t}};\,y_1,\ldots,y_s;\,\frac{zt^q}{(1-4t)^{\rho q}}\right], \tag{2.41}$$

provided that each member of (2.41) *exists.*

We now turn once again to the known finite summation formulas (2.1), (2.15), and (2.17). Indeed, as we have already indicated above, the finite summation formulas (2.15) and (2.17) are substantially the same result: One follows from the other by merely reversing the order of terms in the finite sum and making some obvious variable and parameter changes. In order to show the equivalence of the finite summation formula (2.1) with (2.15) or (2.17), we simply apply the relationship (2.19) appropriately. Thus the three (seemingly different) finite summation formulas (2.1), (2.15), and (2.17) are equivalent to one another.

Although any of the (already proven equivalent) finite summation formulas (2.1), (2.15), and (2.17) does not really fit into the Singhal-Srivastava definition (2.22), yet (just for the sake of completeness) Theorem 7, Theorem 8, and Theorem 9 below can be shown to follow analogously from the finite summation formulas (2.1), (2.15), and (2.17), respectively.

Theorem 7. *Under the applicable hypotheses of Theorem* 3, *let*

$$\Lambda^{(5)}_{n,p,q}[x;\,y_1,\ldots,y_s;\,z] := \sum_{k=0}^{[n/q]} a_k\, C_{n-qk}^{\nu+\rho qk}(x)\,\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k \tag{2.42}$$

$$(a_k \neq 0;\quad n \in \mathbb{N}_0;\quad p,q \in \mathbb{N}),$$

where ρ is a suitable complex parameter. Suppose also that

$$\begin{aligned}
&\mathcal{U}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z) \\
&\quad := \sum_{\ell=0}^{[k/q]} \binom{k-n-2\nu-2\rho q\ell}{k-q\ell} a_\ell\, C_{n-k}^{\nu+\rho q\ell}(x) \\
&\qquad \cdot \Omega_{\mu+p\ell}(y_1, \ldots, y_s)\, z^\ell.
\end{aligned} \tag{2.43}$$

Then

$$\begin{aligned}
&\sum_{k=0}^{n} \mathcal{U}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z)\, t^k \\
&\quad = R^n\, \Lambda_{n,p,q}^{(5)}\left[\frac{x-t}{R}; y_1, \ldots, y_s; z\left(\frac{t}{R}\right)^q\right],
\end{aligned} \tag{2.44}$$

where R is given (as before) with (2.4).

Theorem 8. *Under the applicable hypotheses of Theorem 3, let*

$$\begin{aligned}
&\Lambda_{n,p,q}^{(6)}[x; y_1, \ldots, y_s; z] \\
&\quad := \sum_{k=0}^{[n/q]} a_k\, C_{n-qk}^{\nu+\rho qk}(x)\, \Omega_{\mu+pk}(y_1, \ldots, y_s)\, z^k
\end{aligned} \tag{2.45}$$

$$(a_k \neq 0; \quad n \in \mathbb{N}_0; \quad p, q \in \mathbb{N}),$$

where ρ is a suitable complex parameter. Suppose also that

$$\begin{aligned}
&\mathcal{V}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z) \\
&\quad := \sum_{\ell=0}^{[k/q]} \binom{\nu+k+(\rho-1)q\ell-1}{k-q\ell} a_\ell\, C_{n-k}^{\nu+k+(\rho-1)q\ell}(x) \\
&\qquad \cdot \Omega_{\mu+p\ell}(y_1, \ldots, y_s)\, z^\ell.
\end{aligned} \tag{2.46}$$

Then

$$\begin{aligned}
&\sum_{k=0}^{n} \mathcal{V}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z)\, t^k \\
&\quad = \Lambda_{n,p,q}^{(6)}\left[x + \frac{1}{2}t; y_1, \ldots, y_s; zt^q\right].
\end{aligned} \tag{2.47}$$

Theorem 9. *Under the applicable hypotheses of Theorem 3, let*

$$\begin{aligned}
&\Lambda_{n,p,q}^{(7)}[x; y_1, \ldots, y_s; z] \\
&\quad := \sum_{k=0}^{[n/q]} \binom{n+\rho qk-\nu}{n-qk}^{-1} a_k\, C_{n-qk}^{\nu-n-\rho qk}(x)\, \Omega_{\mu+pk}(y_1, \ldots, y_s)\, z^k
\end{aligned} \tag{2.48}$$

$$(a_k \neq 0; \quad n \in \mathbb{N}_0; \quad p, q \in \mathbb{N}),$$

where ρ is a suitable complex parameter. Suppose also that

$$\mathcal{W}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z)$$
$$:= \sum_{\ell=0}^{[k/q]} \binom{n-q\ell}{n-k} \binom{k+\rho q\ell-\nu}{k-q\ell}^{-1} a_\ell C_{k-q\ell}^{\nu-k-\rho q\ell}(x)$$
$$\cdot \Omega_{\mu+p\ell}(y_1, \ldots, y_s)\, z^\ell. \tag{2.49}$$

Then

$$\sum_{k=0}^{n} \mathcal{W}_{k,p,q}^{\nu,\mu,\rho}(x; y_1, \ldots, y_s; z)\, t^k$$
$$= t^n \Lambda_{n,p,q}^{(7)} \left[x - \frac{1}{2t}; y_1, \ldots, y_s; z \right]. \tag{2.50}$$

Remark 1. The special case of Theorem 3 when $\rho = 0$ is precisely the aforementioned class of bilateral generating functions of Srivastava [20, Part I, p. 229, Corollary 4] (see also [23, p. 422, Corollary 4]). This result of Srivastava [20, Part I, p. 229, Corollary 4] can indeed be shown to yield, as its obvious *further* special cases, numerous families of bilateral generating functions, which are scattered all over the subsequent mathematical investigations (see, for details, González *et al.* [9, p. 166 *et seq.*]).

Remark 2. Various known special cases of some of the other results of this section include special cases of Theorems 4, 5, and 6 when (for example)

$$p = q = 1, \quad \rho = 0, \quad \text{and} \quad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \tag{2.51}$$

or

$$p = q = \rho = 1 \qquad \text{and} \qquad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \tag{2.52}$$

or

$$q = \rho = 1 \qquad \text{and} \qquad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \tag{2.53}$$

or

$$q = 1, \quad \rho = 0, \quad \text{and} \quad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \tag{2.54}$$

were also rederived in many recent investigations (see, for details, González *et al.* [9, p. 166 *et seq.*]). If, in the special case listed under (2.52), we further set $m = 0$, Theorem 4 would immediately yield Theorem 2, that is, the duly-corrected version of the *erroneous* result asserted by Theorem 1.

Remark 3. Since the generating functions (2.2) and (2.14) are equivalent, as we observed above by using the relationship (2.19), their consequences (Theorem 3 and Theorem 4) are also equivalent. Furthermore, since (2.14) itself is a special case of (2.12), Theorem 4 (and hence also Theorem 3) can alternatively be deduced from a result of Srivastava and Popov [24, p. 178, Theorem] by setting $\rho = \sigma$ (see also Srivastava and Handa [22] for *further* extensions involving a general sequence of functions defined by a Rodrigues formula).

Interconnections of some of the other results of this section can also be established by merely examining the generating functions or the finite summation formulas which actually lead us to these results (see, for details, González *et al.* [9, p. 167 *et seq.*]).

3. Generating Functions for Hypergeometric Polynomials

In a recent paper published in this *Revista*, Mukherjee [14] considered the following very special case of the Gauss hypergeometric polynomial defined by (1.12):

$$ {}_2F_1\left(-N,n;\nu+n;x\right)=\sum_{k=0}^{N}\frac{(-N)_k\,(n)_k}{(\nu+n)_k}\,\frac{x^k}{k!} \tag{3.1}$$

$$\left(n,N\in\mathbb{N}_0;\ \nu\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),$$

where, as usual, $\mathbb{N}_0$ and $\mathbb{Z}_0^-$ denote the sets of *nonnegative* and *nonpositive* integers, respectively. By suitably interpreting the parameter n, instead of N (which is done generally [23, Chapter 6, Section 6.7]), Mukherjee [14] derived several bilateral and mixed trilateral generating relations for the *special* Gauss hypergeometric polynomial (3.1) from the group-theoretic (Lie algebraic) viewpoint.

The *main* results of Mukherjee [14] are being recalled here as Theorem 9 and Theorem 10 below.

Theorem 9 (*cf.* [14, p. 123, Theorem 1]). *If there exists a generating relation of the form:*

$$G\left(x,z\right)=\sum_{n=0}^{\infty}a_n\,{}_2F_1\left(-N,n;\nu+n;x\right)z^n, \tag{3.2}$$

then

$$G\left(\frac{x+t}{1+t},\frac{zt}{1+t}\right)=\sum_{n=0}^{\infty}{}_2F_1\left(-N,n;\nu+n;x\right)\rho_n\left(z\right)t^n, \tag{3.3}$$

where

$$\rho_n\left(z\right):=\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\,\frac{(k)_{n-k}\,(\nu+N+k)_{n-k}}{(\nu+k)_{n-k}}\,a_k\,z^k. \tag{3.4}$$

Theorem 10 (*cf.* [14, p. 124, Theorem 2]). *If there exists a generating relation of the form:*

$$H\left(x,y,z\right)=\sum_{n=0}^{\infty}a_n\,{}_2F_1\left(-N,n;\nu+n;x\right)g_n\left(y\right)z^n, \tag{3.5}$$

then

$$H\left(\frac{x+t}{1+t},y,\frac{zt}{1+t}\right)=\sum_{n=0}^{\infty}{}_2F_1\left(-N,n;\nu+n;x\right)\sigma_n\left(y,z\right)t^n, \tag{3.6}$$

where

$$\sigma_n\left(y,z\right):=\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\,\frac{(k)_{n-k}\,(\nu+N+k)_{n-k}}{(\nu+k)_{n-k}}\,a_k\,g_k\left(y\right)z^k. \tag{3.7}$$

Remark 4. Obviously, since the (identically nonvanishing) coefficients a_n $(n\in\mathbb{N}_0)$ are essentially arbitrary, Theorem 10 can, in fact, be deduced from Theorem 9 itself by appropriately letting

$$a_n\longmapsto a_n\,g_n\left(y\right)\qquad\left(n\in\mathbb{N}_0\right) \tag{3.8}$$

in (3.2) and (3.4). Theorem 9, on the other hand, follows from Theorem 10 when we set

$$g_n(y) \equiv 1 \qquad (n \in \mathbb{N}_0) \tag{3.9}$$

or when we let

$$a_n \longmapsto \frac{a_n}{g_n(y)} \qquad (n \in \mathbb{N}_0). \tag{3.10}$$

However, with a view to applying such results as (3.6) above in order to derive bilinear and bilateral generating relations for *simpler* special functions and polynomials, it is usually found to be convenient to specialize a_n and $g_n(y)$ *individually* as well as *separately* (and in a manner dictated by the problem). Thus, for the sake of such conveniences in applying these results, we choose to state the following unification *and* generalization of Theorem 9 and Theorem 10 in the form of the mixed *multilateral* generating relations given by Srivastava [21, p. 3, Theorem 3] (see also Theorems 3 to 8 of Section 2).

Theorem 11. *Corresponding to an identically nonvanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of s (real or complex) variables $y_1,\ldots,y_s$ $(s \in \mathbb{N})$ and of (complex) order μ, let*

$$\Lambda_{\mu,p}[x;y_1,\ldots,y_s;z] := \sum_{n=0}^{\infty} a_n\, {}_2F_1(\lambda,\gamma+n;\nu+n;x) \cdot \Omega_{\mu+pn}(y_1,\ldots,y_s)\, z^n \tag{3.11}$$

$$(a_k \neq 0;\ k \in \mathbb{N}_0;\ p \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}),$$

where λ and γ are suitable complex parameters. Suppose also that

$$\Phi_{n,\mu,p}(y_1,\ldots,y_s;z) := \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \frac{(\gamma+k)_{n-k}(\nu-\lambda+k)_{n-k}}{(\nu+k)_{n-k}} \cdot a_k\, \Omega_{\mu+pk}(y_1,\ldots,y_s)\, z^k. \tag{3.12}$$

Then

$$\sum_{n=0}^{\infty} {}_2F_1(\lambda,\gamma+n;\nu+n;x)\, \Phi_{n,\mu,p}(y_1,\ldots,y_s;z)\, t^n = (1+t)^{-\gamma}\, \Lambda_{\mu,p}\left[\frac{x+t}{1+t};y_1,\ldots,y_s;\frac{zt}{1+t}\right] \qquad (|t|<1), \tag{3.13}$$

provided that each member of (3.13) *exists.*

Proof. A *direct* proof of the general result (3.13) is based upon the following well-known hypergeometric reduction formula [1, p. 24, Equation (28)] (see also [23, p. 105, Equation 2.3 (6)]):

$$F_1[\alpha,\beta,\beta';\beta+\beta';x,y] = (1-y)^{-\alpha}\, {}_2F_1\left(\alpha,\beta;\beta+\beta';\frac{x-y}{1-y}\right) \tag{3.14}$$

$$\left(|y|<1;\ \alpha,\beta,\beta' \in \mathbb{C};\ \beta+\beta' \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where F_1 denotes one of the four Appell functions defined by [1, p. 14, Equation (11)]

$$F_1\left[\alpha,\beta,\beta';\gamma;x,y\right] := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} \tag{3.15}$$

$$\left(\max\{|x|,|y|\}<1;\ \alpha,\beta,\beta'\in\mathbb{C};\ \gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

The details involved are being omitted here (*cf.* [21, pp. 4-5]).

Remark 5. It is easily observed that, in its special case when

$$\lambda=-N \qquad (N\in\mathbb{N}_0), \qquad \gamma=0, \qquad \text{and} \qquad p=s=1 \qquad (y_1=y),$$

Theorem 11 would correspond essentially to Theorem 10. And, just as we pointed out in Remark 4 above, Theorem 9 differs from Theorem 10 *only* notationally in view of (3.8) and (3.10).

Next, by suitably interpreting the parameter α, instead of n (which is done usually (see [23, Chapter 6, Section 6.7]), Mukherjee [16] derived a family of mixed trilateral generating relations for the Gauss hypergeometric polynomial (1) from the viewpoint of Lie groups and Lie algebras. We begin by recalling here the *main* result of Mukherjee [16] in the following (*slightly modified*) form:

Theorem 12 (*cf.* [16, p. 57, Theorem]). *If there exists a generating relation of the following form:*

$$G(x,y;z)=\sum_{n=0}^{\infty} a_n\, {}_2F_1(-n-r,\alpha;\nu;x)\, g_n(y)\, z^n, \tag{3.16}$$

then

$$(1-t)^{\nu-1}(1-xt)^{-\alpha}\ G\left(\frac{x(1-t)}{1-xt},y;zt\right)$$

$$=\sum_{n=0}^{\infty}\rho_n(x,y;z)\,t^n, \tag{3.16}$$

where

$$\rho_n(x,y;z):=\sum_{k=0}^{n} a_k\ \frac{(1-\nu)_{n-k}}{(n-k)!}\ {}_2F_1(-n-r,\alpha;\nu-n+k;x)\, g_k(y)\, z^k. \tag{3.17}$$

In an *earlier* work on this same subject, Chongdar and Pan [6] derived the following *equivalent* version of Theorem 12 when (see also Remark 4 above)

$$g_n(y)\equiv 1 \qquad (n\in\mathbb{N}_0). \tag{3.18}$$

Theorem 13 (*cf.* [6, p. 217, Theorem 8]). *If there exists a generating relation of the form:*

$$F(x,z)=\sum_{n=0}^{\infty} a_n\, {}_2F_1(-n-r,\alpha;\nu;x)\, z^n, \tag{3.19}$$

then

$$(1-t)^{\nu-1}(1-xt)^{-\alpha}\ F\left(\frac{x(1-t)}{1-xt}, zt\right)$$
$$= \sum_{n=0}^{\infty} \sigma_n(x,z)\, t^n, \tag{3.20}$$

where

$$\sigma_n(x,z) := \sum_{k=0}^{n} a_k \frac{(1-\nu)_{n-k}}{(n-k)!}\ {}_2F_1(-n-r,\alpha;\nu-n+k;x)\, z^k. \tag{3.21}$$

Remark 6. Just as we observed above about the equivalence of Theorem 9 and Theorem 10 (see Remark 4), Theorem 12 and Theorem 13 are *also* equivalent.

Remark 7. An obviously *erroneous* version of a *special* case of Theorem 12 when $r = 0$ was claimed to be a *new* result in a very recent paper by Chongdar and Sen [7, p. 85, Corollary 7].

An interesting unification (and generalization) of the *equivalent* results (Theorem 12 and Theorem 13) is provided by Theorem 14 below (*cf.* Lin *et al.* [11, p. 611, Theorem 3]).

Theorem 14. *Corresponding to an identically nonvanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of s (real or complex) variables $y_1,\ldots,y_s$ ($s \in \mathbb{N}$) and of (complex) order μ, let*

$$\Lambda_\mu^{(p,q)}[x;y_1,\ldots,y_s;z] := \sum_{n=0}^{\infty} a_n\, {}_2F_1(\rho-qn,\alpha;1-\lambda;x)$$
$$\cdot\, \Omega_{\mu+pn}(y_1,\ldots,y_s)\, z^n \tag{3.23}$$
$$(a_k \neq 0;\ k \in \mathbb{N}_0;\ p,q \in \mathbb{N};\ \lambda \in \mathbb{C}\setminus\mathbb{N}),$$

where ρ and α are suitable complex parameters. Suppose also that

$$\omega_n(x;y_1,\ldots,y_s;z) := \sum_{k=0}^{[n/q]} a_k \frac{(\lambda)_{n-qk}}{(n-qk)!}\ {}_2F_1(\rho-n,\alpha;1-\lambda-n+qk;x)$$
$$\cdot\, \Omega_{\mu+pk}(y_1,\ldots,y_s)\, z^k. \tag{3.24}$$

Then

$$\sum_{n=0}^{\infty} \omega_n(x;y_1,\ldots,y_s;z)\, t^n = (1-t)^{-\lambda}(1-xt)^{-\alpha}$$
$$\cdot\, \Lambda_\mu^{(p,q)}\left[\frac{x(1-t)}{1-xt}; y_1,\ldots,y_s; zt^q\right] \tag{3.25}$$
$$\left(|t| < \min\left\{1, |x|^{-1}\right\}\right),$$

provided that each member of (3.25) *exists.*

Proof. In the *direct* proof of Theorem 14, *without* using the group-theoretic (Lie algebraic) technique employed by Mukherjee [16] in his rederivation of the very specialized cases given by

Theorem 12 and Theorem 13 above, one would require the following hypergeometric generating function [11, p. 612, Equation (18)]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\rho - n, \alpha; 1-\lambda-n; x)\, t^n$$
$$= (1-t)^{-\lambda}(1-xt)^{-\alpha}\, {}_2F_1\left(\rho, \alpha; 1-\lambda; \frac{x(1-t)}{1-xt}\right) \tag{3.26}$$
$$\left(|t| < \min\left\{1, |x|^{-1}\right\};\ \rho, \alpha \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \mathbb{N}\right),$$

which Lin *et al.* [11] derived as one of the two special cases of a general family of hypergeometric generating functions proven by them [11, p. 612, Equation (16)]. The details involved may be omitted here.

Remark 8. It is easily seen that, in its special case when

$$\rho = -r \quad (r \in \mathbb{N}_0), \quad \lambda = 1-\nu, \quad \text{and} \quad p = q = s = 1 \quad (y_1 = y),$$

Theorem 14 would reduce immediately to Theorem 12. And, just as we pointed out in Remark 8, Theorem 13 differs from Theorem 12 only notationally in view of (3.9) and (3.10).

In the theory of generating functions (see, for details, [23]), various families of generalized hypergeometric polynomials in one, two and more variables have also been investigated (see, for example, [2], [3] and [10]). In particular, we recall here the following well-known (rather classical) result [23, p. 138, Equation 2.6 (8)]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{u+1}F_v\left[\begin{array}{l} -n, \alpha_1, \ldots, \alpha_u; \\ \beta_1, \ldots, \beta_v; \end{array} x\right] t^n$$
$$= (1-t)^{-\lambda}\, {}_{u+1}F_v\left[\begin{array}{l} \lambda, \alpha_1, \ldots, \alpha_u; \\ \beta_1, \ldots, \beta_v; \end{array} -\frac{xt}{1-t}\right] \tag{3.27}$$
$$\left(|t| < 1;\ \lambda, \alpha_j \in \mathbb{C}\ (j = 1, \ldots, u);\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1, \ldots, v)\right).$$

Now, since [23, p. 42, Equation 1.4 (3)]

$${}_{u+1}F_v\left[\begin{array}{l} -n, \alpha_1, \ldots, \alpha_u; \\ \beta_1, \ldots, \beta_v; \end{array} z\right] = \frac{(\alpha_1)_n \cdots (\alpha_u)_n}{(\beta_1)_n \cdots (\beta_v)_n} (-z)^n$$
$$\cdot\ {}_{v+1}F_u\left[\begin{array}{l} -n, 1-\beta_1-n, \ldots, 1-\beta_v-n; \\ 1-\alpha_1-n, \ldots, 1-\alpha_u-n; \end{array} \frac{(-1)^{u+v}}{z}\right], \tag{3.28}$$

where we have simply reversed the order of the terms of the generalized hypergeometric polynomials occurring on the left-hand side, it is fairly straightforward to rewrite the well-known (rather classical) result (3.27) in the following *equivalent* form:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n (\alpha_1)_n \cdots (\alpha_u)_n}{(\beta_1)_n \cdots (\beta_v)_n} \, {}_{v+1}F_u \left[\begin{array}{c} -n, 1-\beta_1-n, \ldots, 1-\beta_v-n; \\ \\ 1-\alpha_1-n, \ldots, 1-\alpha_u-n; \end{array} x \right] \frac{t^n}{n!}$$

$$= \left[1+(-1)^{u+v}\, xt\right]^{-\lambda} \, {}_{u+1}F_v \left[\begin{array}{c} \lambda, \alpha_1, \ldots, \alpha_u; \\ \\ \beta_1, \ldots, \beta_v; \end{array} \frac{t}{1+(-1)^{u+v}\, xt} \right], \tag{3.29}$$

which follows also by suitably specializing some general families of generating functions derived during the 1960s and 1970s by Srivastava (*cf.*, *e. g.*, [23, p. 145, Equation 2.6 (30)], it being tacitly assumed that both sides of (3.29) exist.

In a very specialized case of this last generating function (3.29) when

$$u = v = 1 \qquad (\alpha_1 = 1-\rho;\ \beta_1 = 1-\alpha = \lambda),$$

we easily find for the Gauss hypergeometric polynomials that [*cf.* Equation (3.26)]

$$\sum_{n=0}^{\infty} \frac{(1-\rho)_n}{n!} \, {}_2F_1(-n, \alpha-n; \rho-n; x)\, t^n = (1+xt)^{\alpha-\rho}\left[1-(1-x)t\right]^{\rho-1}. \tag{3.30}$$

Two *equivalent* results involving the Gauss hypergeometric polynomials occurring in (3.30) were asserted *incorrectly* by Mukherjee [15, p. 255, Theorems 1 and 2], each of which do not seem to make any sense whatsoever. Here we make use of the substantially more general result (3.29) with a view to deriving the following companion of (for example) Theorem 14.

Theorem 15. *Under the hypotheses of Theorem* 14, *let*

$$\Xi_{\mu}^{(p,q)}[x; y_1, \ldots, y_s; z] := \sum_{n=0}^{\infty} a_n \, \Omega_{\mu+pn}(y_1, \ldots, y_s)$$

$$\cdot \, {}_{u+1}F_v \left[\begin{array}{c} \lambda+qn, \alpha_1+qn, \ldots, \alpha_u+qn; \\ \\ \beta_1+qn, \ldots, \beta_v+qn; \end{array} x \right] z^n \tag{3.31}$$

$$(a_k \neq 0;\ k \in \mathbb{N}_0;\ p, q \in \mathbb{N}),$$

where λ, α_j $(j = 1, \ldots, u)$, *and* β_j $(j = 1, \ldots, v)$ *are suitably constrained complex parameters. Suppose also that*

$$\kappa_n(x; y_1, \ldots, y_s; z) := \sum_{k=0}^{[n/q]} a_k \, \Omega_{\mu+pk}(y_1, \ldots, y_s) \cdot \frac{(\lambda + qk)_{n-qk} (\alpha_1 + qk)_{n-qk} \cdots (\alpha_u + qk)_{n-qk}}{(n-qk)! \, (\beta_1 + qk)_{n-qk} \cdots (\beta_v + qk)_{n-qk}} \cdot {}_{v+1}F_u \left[\begin{matrix} -n + qk, 1 - \beta_1 - n, \ldots, 1 - \beta_v - n; \\ 1 - \alpha_1 - n, \ldots, 1 - \alpha_u - n; \end{matrix} \; x \right] z^k. \tag{3.32}$$

Then

$$\sum_{n=0}^{\infty} \kappa_n(x; y_1, \ldots, y_s; z) \, t^n = \left[1 + (-1)^{u+v} \, xt\right]^{-\lambda} \cdot \Xi_{\mu}^{(p,q)} \left[\frac{t}{1 + (-1)^{u+v} \, xt}; y_1, \ldots, y_s; \frac{zt^q}{\left[1 + (-1)^{u+v} \, xt\right]^q} \right] \tag{3.33}$$

$$\left(|t| < |x|^{-1}\right),$$

provided that each member of (3.33) *exists.*

We conclude our present investigation by remarking that, in several recent works (see, for example, [5] and [7]), the existence of some generating functions for the product of two Gauss hypergeometric polynomials was claimed. In particular, the following *bilinear* sum was involved in Chongdar's work [5, p. 120, Theorem 7]:

$$\sum_{n=0}^{\infty} a_n \, {}_2F_1(-n, \beta; \nu; x) \, {}_2F_1(-n, \beta; n; x) w^n,$$

which is obviously *meaningless*, since the *denominator* parameter n of the *second* hypergeometric polynomial ought to be restricted by [*cf.* Equation (1.12)]

$$n \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}.$$

Acknowledgements

The present investigation was supported, in part, by the *National Science Council of the Republic of China* under Grant NSC-94-2115-M-033-004, the *Faculty Research Program of Chung Yuan Christian University* under Grant CYCU-94RD-RA001-10108, and the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

[1] P. Appell *et* J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] L. Carlitz and H. M. Srivastava, Some hypergeometric polynomials associated with the Lauricella function F_D of several variables. I and II, *Mat. Vesnik* **13** **(28)** (1976), 41-47 and 143-152.

[3] W.-C. C. Chan, K.-Y. Chen and H. M. Srivastava, Certain classes of generating functions for the Jacobi and related hypergeometric polynomials, *Comput. Math. Appl.* **44** (2002), 1539-1556.

[4] K.-Y. Chen and H. M. Srivastava, A new result for hypergeometric polynomials, *Proc. Amer. Math. Soc.* **133** (2005), 3295-3302.

[5] A. K. Chongdar, Unified theory of obtaining a more general class of generating relations from a given class of quasi-bilateral generating relations involving some special functions by group theoretic method, *Rev. Acad. Canaria Cienc.* **11** (1999), 111-121.

[6] A. K. Chongdar and S. K. Pan, On a class of bilateral generating functions, *Simon Stevin* **66** (1992), 207-220.

[7] A. K. Chongdar and B. K. Sen, Group-theoretic method of obtaining a class of mixed trilateral generating relations for certain special functions, *Rev. Acad. Canaria Cienc.* **16** (2004), 71-87.

[8] K. Driver and M. Möller, Quadratic and cubic transformations and zeros of hypergeometric polynomials, *J. Comput. Appl. Math.* **142** (2002), 411-417.

[9] B. González, J. Matera and H. M. Srivastava, Some q-generating functions and associated generalized hypergeometric polynomials, *Math. Comput. Modelling* **34** (1-2) (2001), 133-175.

[10] S.-D. Lin, Y.-S. Chao and H. M. Srivastava, Some families of hypergeometric polynomials and associated integral representations, *J. Math. Anal. Appl.* **294** (2004), 399-411.

[11] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some mixed multilateral generating relations involving hypergeometric functions, *Integral Transform. Spec. Funct.* **18** (2005), 609-614.

[12] A. B. Majumdar, On bilateral generating functions. II, *Rev. Acad. Canaria Cienc.* **7** (1995), 111-115.

[13] E. B. McBride, *Obtaining Generating Functions*, Springer Tracts in Natural Philosophy, Vol. **21**, Springer-Verlag, New York, Heidelberg and Berlin, 1971.

[14] M. C. Mukherjee, On some generating functions of hypergeometric polynomial. II, *Rev. Acad. Canaria Cienc.* **11** (1999), 123-126.

[15] M. C. Mukherjee, On some generating functions of modified hypergeometric polynomial. I, *Acta Cienc. Indica (Math.)* **27** (2001), 255-257.

[16] M. C. Mukherjee, On extension of mixed trilateral generating function of hypergeometric polynomials, *Gaṇita* **53** (2002), 57-59.

[17] M. C. Mukherjee, On partial semi bilinear generating function involving hypergeometric polynomial. I, *Rev. Acad. Canaria Cienc.* **14** (2002), 93-96.

[18] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[19] J. P. Singhal and H. M. Srivastava, A class of bilateral generating functions for certain classical polynomials, *Pacific J. Math.* **42** (1972), 755-762.

[20] H. M. Srivastava, Some bilateral generating functions for a certain class of special functions. I and II, *Nederl. Akad. Wetensch. Indag. Math.* **42** (1980), 221-233 and 234-246.

[21] H. M. Srivastava, Some families of mixed multilateral generating relations, *J. Rajasthan Acad. Phys. Sci.* **1** (2002), 1-6.

[22] H. M. Srivastava and S. Handa, Some general theorems on generating functions and their applications, *Appl. Math. Lett.* 1 (4) (1988), 391-394.

[23] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[24] H. M. Srivastava and B. S. Popov, A class of generating functions for the Jacobi and related polynomials, *Stud. Appl. Math.* **78** (1988), 177-182.

[25] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. **23**, Fourth Edition, American Mathematical Society, Providence, Rhode Island, 1975.