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ANALYTIC CHARACTERIZATIONS OF ANNULI

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ABSTRACT

Characterizations of annuli among plane domains with analytic boundary in terms of potential theory and of quadrature identities are obtained using elementary techniques.

KEYWORDS: Cauchy problem, harmonic measure, Poisson equation, quadrature identity.

I. INTRODUCTION AND MAIN RESULT

Let G be a finitely-connected plane domain whose boundary Γ consists of two or more pairwise disjoint, analytic closed curves. Denote by γ_0 the "outer" boundary component, and write $\gamma_1=\Gamma-\gamma_0$. Also, if A_0, P_0 and A_1, P_1 represent the area and perimeter of the domains enclosed by γ_0 and γ_1, respectively, then set $\mu_0=A_0/P_0$, $\mu_1= -A_1/P_1$. The aim of this note is to prove the following result.

<u>Theorem 1</u>: *In the above notation and hypotheses, let further* dA *represent the area element in* G, *and let* s *denote the arclength parameter on* Γ. *Then, the following are equivalent:*

(i) The Cauchy problem

$$\Delta v \equiv -1 \quad \text{in } G,$$

$$v= -\mu_0^2 \quad \text{on } \gamma_0, \qquad v= -\mu_1^2 \quad \text{on } \gamma_1,$$

$$\frac{\partial v}{\partial n} = -\mu_0 \quad \text{on } \gamma_0, \qquad \frac{\partial v}{\partial n} = -\mu_1 \quad \text{on } \gamma_1$$

is solvable in G. Here, $\frac{\partial v}{\partial n}$ *denotes the outer normal derivative of* v *on* Γ.

(ii) The quadrature identity

$$\iint_G u\, dA = \mu_0 \int_{\gamma_0} u\, ds + \mu_1 \int_{\gamma_1} u\, ds - \mu_0^2 \int_{\gamma_0} \frac{\partial u}{\partial n}\, ds - \mu_1^2 \int_{\gamma_1} \frac{\partial u}{\partial n}\, ds \qquad (1)$$

holds for every harmonic function u *in* G.

(iii) G *is an annulus centered at the origin with outer radius* $2\mu_0$ *and inner radius* $-2\mu_1$.

We recall that a *quadrature identity* for a vector space F of functions defined and integrable with respect to area measure in the plane domain Ω is an identity of the type

$$\iint_G f\, dA = L(f), \qquad (2)$$

valid for all functions f in the given class F, where L is some linear functional on F. See, e. g., [1], [2] and references therein for further orientation on this vaste subject.

Another characterization of the annulus by a quadrature identity has been obtained by Avci [3]. A condition analogous to the one in part (i) of Theorem 1, also requiring the solvability of certain (overdetermined) Cauchy problem for a Poisson differential equation, has been shown to characterize the balls in the Euclidean space $\mathbb{R}^n$ by Serrin [4]. The connection between quadrature identities and differential equations has been pointed out by Shapiro [5]. It should be remarked, however, that the techniques used in [3] and [5] are not suitable to deal with quadrature identities (2) where the defining functional L is supported on the boundary of Ω, as it happens in (1). Further, the methods chosen in our approach are simple and elementary.

In the next Section II we present an auxiliary result, needed in the proof of Theorem 1, which will be deferred until the last Section III. Throughout the rest of this note, r will always stand for the distance from the origin in $\mathbb{C}$.

II. AN AUXILIARY LEMMA

The proof of Theorem 1 is based on the following observation due to Weinberger [6], which we label as Lemma 2.

Lemma 2: *Let the real-valued functions* t, v *satisfy the Poisson equations* $\Delta t \equiv -1$, $\Delta v \equiv -1$ *in the plane domain* G. *Then,*

$$w = |\mathrm{grad}\ v|^2 + t$$

is subharmonic in G. *Moreover,* w *is harmonic if, and only if,*

$$v = -\frac{1}{4} r^2 + c \tag{3}$$

for some real constant c.

Proof: Since

$$1 = (\Delta v)^2 \leq 2 \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 \right] \tag{4}$$

$$\leq 2 \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x\, \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 \right] \tag{5}$$

$$= \Delta |\mathrm{grad}\ v|^2,$$

we have

$$\Delta w = \Delta |\mathrm{grad}\ v|^2 - 1 \geq 0, \tag{6}$$

thus proving that w is subharmonic in G. If w is harmonic then equality holds in (6). This leads to an equality also in (4) and (5). As $\Delta v \equiv -1$, it follows that

$$\frac{\partial^2 v}{\partial x^2} \equiv \frac{\partial^2 v}{\partial y^2} \equiv -\frac{1}{2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x\, \partial y} \equiv 0$$

in G. Hence, $v= -\frac{1}{4}r^2$ up to an additive real constant. The converse, that w is harmonic when v has the form (3) and $\Delta t \equiv -1$, is clear.

III. PROOF OF THEOREM 1

We begin by noticing that if (iii) holds then (3), with $c=0$, is a solution of the Cauchy problem (i). Thus, we only need to show that (i) is equivalent to (ii), and that (i) implies (iii).

(i) implies (ii): Let v be as stated in (i), and let u be harmonic in G. Then, from Green's formula it follows that

$$\begin{aligned}\iint_G u\, dA &= -\iint_G u\, \Delta v\, dA \\ &= -\iint_G v\, \Delta u\, dA + \int_\Gamma v \frac{\partial u}{\partial n} ds - \int_\Gamma u \frac{\partial v}{\partial n} ds \\ &= \int_\Gamma v \frac{\partial u}{\partial n} ds - \int_\Gamma u \frac{\partial v}{\partial n} ds \\ &= \mu_0 \int_{\gamma_0} u\, ds + \mu_1 \int_{\gamma_1} u\, ds - \mu_0^2 \int_{\gamma_0} \frac{\partial u}{\partial n} ds - \mu_1^2 \int_{\gamma_1} \frac{\partial u}{\partial n} ds .\end{aligned}$$

This establishes (ii).

(ii) implies (i): Set $t=u_0-\frac{1}{4}r^2$, where u_0 is the solution of Dirichlet's problem in G with boundary data $\frac{1}{4}r^2$. Note that t satisfies the Poisson equation $\Delta t \equiv -1$ in G and is zero on Γ. Next, let u be an arbitrary C^∞ function on Γ, and denote also by u its harmonic extension into G. By Green's Formula,

$$\begin{aligned}\int_\Gamma u \frac{\partial t}{\partial n} ds &= -\iint_G u\, dA \\ &= -\int_\Gamma (\mu_0\omega_0 + \mu_1\omega_1 - \mu_0^2 \frac{\partial \omega_0}{\partial n} - \mu_1^2 \frac{\partial \omega_1}{\partial n})\, u\, ds,\end{aligned}$$

where ω_0, ω_1 are the harmonic measures of γ_0, γ_1, respectively (that is, w_i is the solution of Dirichlet's problem in G with boundary data 1 on γ_i, 0 on $\Gamma-\gamma_i$ (i=0, 1)). Now, being u arbitrary, we conclude that

$$\frac{\partial t}{\partial n} = -(\mu_0\omega_0 + \mu_1\omega_1) + \frac{\partial}{\partial n}(\mu_0^2\omega_0 + \mu_1^2\omega_1)$$

The function $v = t-\mu_0^2\omega_0 - \mu_1^2\omega_1$ is thus a solution of the Cauchy problem (i).

<u>(i) implies (iii)</u>: It suffices to show that v has the form (3) for some real constant c. This will follow at once from Lemma 2 if we can prove that

$$w = |\text{grad } v|^2 + t$$

is harmonic in G, where t is as described above. With this purpose we define

$$h = \mu_0^2\omega_0 + \mu_1^2\omega_1.$$

The function h is harmonic in G and has the same boundary values as w. Since w is subharmonic (Lemma 2), by the Maximum Principle either

$$w < h \quad \text{in } G \tag{7}$$

or

$$w \equiv h \quad \text{in } G. \tag{8}$$

Nonoccurrence of (7) would complete the proof of Theorem 1. But (7) cannot hold, since

$$\iint_G w\, dA = \iint_G h\, dA, \tag{9}$$

as we now proceed to show. Indeed, observing that t=v+h and using Green's formula, the first member of (9) is computed to be

$$2\iint_G v\,dA + \int_\Gamma v\,\frac{\partial v}{\partial n}\,ds + \iint_G h\,dA \tag{10}$$

Another application of Green's Formula then yields

$$\begin{aligned}
2\iint_G v\,dA &= \int_\Gamma v\,\frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right)ds - \iint_G r\,\frac{\partial v}{\partial r}\,dA \\
&= \int_\Gamma v\,\frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right)ds + \int_\Gamma \left(\frac{\partial v}{\partial n}\right)^2 \frac{\partial}{\partial n}\left(\frac{1}{4}r^2\right)ds \\
&= -\mu_0^2 A_0 + \mu_1^2 A_1 .
\end{aligned} \tag{11}$$

But, on the other hand,

$$\begin{aligned}
\int_\Gamma v\,\frac{\partial v}{\partial n}\,ds &= \mu_0^3 P_0 + \mu_1^3 P_1 \\
&= \mu_0^2 A_0 - \mu_1^2 A_1 .
\end{aligned} \tag{12}$$

Insertion of (11) and (12) into (10) proves (9) and the Theorem.

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