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NOTE ON THE HAUSDORFF MEASURE OF NONCOMPACTNESS [1]

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ABSTRACT

A result of R. Harte and A. Wickstead for Banach spaces is generalized in this note for families on a complete metric space in which is defined a set measure: if (x_i) is a bounded family, then the set measure of its range $\{x_i\}$ agrees with the distance of (x_i) to the set of all families with range of null set measure, in the the metric space of bounded families.

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INTRODUCTION

B.N. Sadovskii [SA] considers for a Banach space X the quotient of the space of all bounded sequences in X, $\ell_\infty(X)$, by the subspace of all sequences with relatively compact range, m(X). If we denote

$$P(X) := \ell_\infty(X)/m(X) \ ,$$

then the norm of a coset $(x_n)+m(X)$ in P(X) verifies [HW]:

$$\|(x_n)+m(X)\| = h(\{x_n\}) \ ,$$

where $h(\{x_n\})$ is the Hausdorff measure of noncompactness of $\{x_n\}$, the range of (x_n). Independently, P has been defined by J.J. Buoni, R. Harte and T. Wickstead [BHW].

In this note we replace the Banach space X by a complete metric space E, the Hausdorff measure of noncompactness h by a set measure μ, and the sequences (x_n) by families (x_i). We obtain a generalization of above result.

PRELIMINARIES

Let E be complete metric space. We denote its distance by d. For $x\in E$, $\emptyset\neq A\subset E$ and $\varepsilon>0$ a real number, we use the following notation:

$$K(x,\varepsilon) := \{y\in E : d(x,y)<\varepsilon\} \ ,$$

$$K(A,\varepsilon) := \{y\in E : d(y,A)<\varepsilon\} \ ,$$

$$P_b(E) := \{A\neq\emptyset : A\subset E \text{ bounded}\} \ .$$

Moreover, if $A,B\in P_b(E)$ we define the Hausdorff distance between A and B in the following way:

$$D(A,B) := \inf \{\varepsilon>0 : A\subset K(B,\varepsilon) \text{ and } B\subset K(A,\varepsilon)\} \ .$$

1 DEFINITION [MA]

The map μ: $P_b(E) \longrightarrow \mathbb{R}$ is called a **set measure** if

(1) for every $A\in P_b(E)$, $\mu(A)\geq 0$.

(2) there exists $N \in P_b(E)$ such that $\mu(N)=0$.

(3) there exists $r(\mu) \geq 0$ such that if $A \in P_b(E)$ and $\varepsilon>0$, then

$$\mu(K(A,\varepsilon)) \leq \mu(A) + r(\mu)\varepsilon .$$

(4) if $A,B \in P_b(E)$ and $A \subset B$, then $\mu(A) \leq \mu(B)$.

(5) for every $A,B \in P_b(E)$, $\mu(A \cup B) \leq \max \{\mu(A),\mu(B)\}$.

Moreover, the kernel of μ is defined by $Ker(\mu) := \{N \in P_b(E) : \mu(N)=0\}$.

In [MA] it is show that for every set measure μ there exists a **canonical measure** μ_c such that $Ker(\mu)=Ker(\mu_c)$ and

$$\mu_c(A) = D(A,Ker(\mu)) := \inf \{D(A,N) : \mu(N)=\mu_c(N)=0\} .$$

The Hausdorff measure of noncompactness of $A \in P_b(E)$, $h(A)$, is the infimum of the $\varepsilon>0$ such that A can be covered by a finite number of balls of radius less than ε; that is,

$$h(A) := \inf \{\varepsilon>0 : D \text{ has a finite } \varepsilon\text{-net in } E\} .$$

h is a set measure, because it verifies the Definition 1. Moreover h is the canonical measure of kernel the class of all relatively compact subsets.

MAIN RESULT

We consider the set of all bounded families (x_i) in E with index set I:

$$\ell_\infty(I,E) := \{(x_i) \subset E : (x_i) \text{ bounded}\} ,$$

attached with the distance

$$d((x_i),(y_i)) := \sup \{d(x_i,y_i) : i \in I\} .$$

Then $\ell_\infty(I,E)$ is a complete metric space.

If μ is a canonical measure in E, we consider the subset of $\ell_\infty(I,E)$ of the families (x_i) with range $\{x_i\}:=\{x_i : i \in I\}$ belongs to $Ker(\mu)$.

$$\mu(I,E) := \{(x_i) \in \ell_\infty(I,E) : \mu(\{x_i\})=0\} .$$

2 THEOREM

If $(x_i)\in\ell_\infty(I,E)$, then

$$\mu(\{x_i\}) = d((x_i),\mu(I,E)) .$$

PROOF. If $\varepsilon>\mu(\{x_i\})=D(\{x_i\},Ker(\mu))$, then there exists $N\in Ker(\mu)$ such that $D(\{x_i\},N)<\varepsilon$. Hence $\{x_i\}\subset K(N,\varepsilon)$ and, for every $i\in I$, there exists $z_i\in N$ such that $d(x_i,z_i)<\varepsilon$. Hence $(z_i)\in\mu(I,E)$ and $d((x_i),(z_i))<\varepsilon$. Then $d((x_i),\mu(I,E))<\varepsilon$. Consequently

$$d((x_i),\mu(I,E)) \le \mu(\{x_i\}) .$$

If $\varepsilon>d((x_i),\mu(I,E))$, there exists $(z_i)\in\mu(I,E)$ such that $d((x_i),(y_i))<\varepsilon$; hence for every $i\in I$, we obtain $d(x_i,y_i)<\varepsilon$, and consequently $\{x_i\}\subset K(\{z_i\},\varepsilon)$. Then

$$\mu(\{x_i\}) \le \mu(K(\{z_i\},\varepsilon)) \le \mu(\{z_i\}) + \varepsilon = \varepsilon .$$

It follows that $\mu(\{x_i\})\le\varepsilon$. Consequently

$$\mu(\{x_i\}) \le d((x_i),\mu(I,E)) .$$ ■

3 COROLLARY

m(E) is closed in $\ell_\infty(E)$.

If we take E=X (a Banach space), μ=h (the Hausdorff measure of noncompactness) and I=ℕ (natural numbers), then $\ell_\infty(\mathbb{N},X)=\ell_\infty(X)$ and h(ℕ,X)=m(X) (see Introduction) the above results can be write in the following way:

4 COROLLARY [HW]

(1) $h(\{x_n\}) = d((x_n),m(X))$,

(2) m(X) is closed in $\ell_\infty(X)$.

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