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AN ASSOCIATED STRUCTURE OF A MODULE

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Abstract

In this paper we have generalised the concept of a module in the sense that every module can be embedded into this new structure, which we name as 'quasi module', and every quasi module contains a module. In fact, we have replaced the group structure of a module by a semigroup structure and invited a partial order which has a significant role in formulating this new structure; it is this partial order which is the prime key in relating a quasi module with a module. After discussing several examples we have introduced the concept of order-morphism between two quasi modules, discussed its various properties and finally proved an isomorphism theorem regarding this order-morphism.

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1 Introduction

For any topological module M over a topological unitary ring R, the collection $\mathscr{C}(M)$ of all nonempty compact subsets of M is closed under usual addition of two sets and multiplication of a set by any element of R. Also for any $r, s \in R$ and any $A, B \in \mathscr{C}(M)$ with $A \subseteq B$ we have $(r+s)A \subseteq rA + sA$ and $rA \subseteq rB$. Moreover, if θ be the additive identity in M then $A - A = \{\theta\}$ iff A is a singleton set. Thus $\big\{\{m\} : m \in M\big\}$ is the collection of all invertible elements of $\mathscr{C}(M)$, $\{\theta\}$ acting as the additive identity in $\mathscr{C}(M)$. These singletons are the minimal elements of $\mathscr{C}(M)$ with respect to the usual set-inclusion as partial order. Now this collection of all minimal elements of $\mathscr{C}(M)$ can be identified with the module M through the map $\{m\} \longmapsto m$ ($m \in M$). This makes a useful connection between the hyperspace $\mathscr{C}(M)$ and its generating module M. The above facts are not just a speciality of the hyperspace $\mathscr{C}(M)$; we have axiomatised these facts and introduced the concept of a *quasi module*, as explained below.

In this paper we have generalised the concept of a module in the sense that every module can be embedded into this new structure, which we name as 'quasi module', and every quasi module contains a module. In fact, we have replaced the group structure of a module by a semigroup structure and invited a partial order within this structure which has a significant role in formulating this new structure; it is this partial order which is the prime key in relating a quasi module with a module. This partial order is made compatible with the semigroup operation and external composition (which is multiplication by an unitary ring, in this case), while formulating the axiom for quasi module. A number of examples have been discussed and it has been shown that every module over an unitary ring can be embedded into a quasi module and every quasi module contains a module as a sub-structure.

In section 3 we have introduced the concept of an order-morphism between two quasi modules over a common unitary ring. Some of its properties have been discussed. Section 4 deals with the arbitrary product of quasi modules. We have shown that Cartesian product of any family of quasi modules is again a quasi module. After defining the kernel of an order-morphism we have proved that kernel of any order-morphism is a quasi module.

In the last section we have discussed an order-isomorphism theorem. For doing this we have introduced first the concept of congruence in a quasi module and then constructed a quotient structure which has been finally settled as a quasi module.

2 Quasi Module

Definition 2.1. Let $(X, \leq)$ be a partially ordered set, '+' be a binary operation on X and '$\cdot$': $R \times X \longrightarrow X$ be another composition [R being a unitary ring]. If the operations and partial order satisfy the following axioms then $(X, +, \cdot, \leq)$ is called a *quasi module* (in short *qmod*) over R.

$A_1 : (X, +)$ is a commutative semigroup with identity θ.

$A_2 : x \leq y \ (x, y \in X) \Rightarrow x + z \leq y + z, \ r \cdot x \leq r \cdot y, \ \forall z \in X, \forall r \in R.$

$A_3 :$ (i) $r \cdot (x + y) = r \cdot x + r \cdot y,$

(ii) $r \cdot (s \cdot x) = (rs) \cdot x,$

(iii) $(r+s)\cdot x \leq r\cdot x + s\cdot x$,

(iv) $1\cdot x = x$, '1' being the multiplicative identity of R

(v) $0\cdot x = \theta$ and $r\cdot\theta = \theta$ $(r\in R)$

$\forall\, x, y \in X,\ \forall\, r, s \in R$.

$A_4 : x + (-1)\cdot x = \theta$ if and only if $x \in X_0 := \Big\{ z \in X : y \nleq z, \forall\, y \in X \smallsetminus \{z\} \Big\}$

A_5 : For each $x \in X, \exists\, y \in X_0$ such that $y \leq x$.

The elements of the set X_0 (which are evidently the minimal elements of X with respect to the partial order '$\leq$') are called '*one order*' elements of X and, by axiom A_4, these are the *only* invertible elements of X, the inverse of $x\,(\in X_0)$ being $(-1)\cdot x$, usually written as '$-x$'. Also for any $x, y \in X_0$ and $\forall\, r \in R$ we have, by axiom A_4, $x - x = \theta$, $y - y = \theta$ $\Rightarrow (x+y) - (x+y) = \theta$ and $rx - rx = r(x-x) = \theta$ and hence $rx, x+y \in X_0$. Moreover, for $r, s \in R$ and $x \in X_0$ we have $(r+s)x \leq rx + sx \Rightarrow (r+s)x = rx + sx$ ($\because rx + sx$ is of order one). Thus we have the following result.

Result 2.2. *For any quasi module X over an unitary ring R, the set X_0 of all one order elements of X is a module over R.*

Above result shows that every quasi module contains a module. It is now a routine work to verify that, for a topological module M over a topological unitary ring R, the collection $\mathscr{C}(M)$ of all nonempty compact subsets of M forms a quasi module over R with usual set-inclusion as partial order and the relevent operations defined as follows : for $A, B \in \mathscr{C}(M)$ and $r \in R$, $A + B := \{a + b : a \in A, b \in B\}$ and $rA := \{ra : a \in A\}$. The identity element of $\mathscr{C}(M)$ is $\{\theta\}$, where θ is the identity element of M; the set of all one order elements of $\mathscr{C}(M)$ is given by $[\mathscr{C}(M)]_0 = \Big\{\{m\} : m \in M\Big\}$. If we identify $\Big\{\{m\} : m \in M\Big\}$ with M we can say that, the topological module M is embedded into the quasi module $\mathscr{C}(M)$. We construct below an example which shows that every module (not necessarily topological) over an unitary ring can be embedded into a quasi module over the same ring.

Example 2.3. Let M be a module over an unitary ring R. Let $\widetilde{M} := M \bigcup \{\omega\}$ $(\omega \notin M)$. Define '+', '$\cdot$' and the partial order '$\leq_p$' as follows :

(i) The operation '+' between any two elements of M is same as in the module M and $x + \omega := \omega$ and $\omega + x := \omega$, $\forall\, x \in \widetilde{M}$.

(ii) The operation '$\cdot$' when applied on $R \times M$ is same as in the module M and $r\cdot\omega := \omega$, if $r(\neq 0) \in R$ and $0\cdot\omega := \theta$, θ being the identity element in M.

(iii) $x \leq_p \omega$, $\forall\, x \in M$ and $x \leq_p x$, $\forall\, x \in \widetilde{M}$.

(i) The operation '+' between any two elements of M is same as in the module M and $x+\omega := \omega$ and $\omega + x := \omega$, $\forall x \in \widetilde{M}$.

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(iii) $x \leq_p \omega$, $\forall x \in M$ and $x \leq_p x$, $\forall x \in \widetilde{M}$.

We show that $\left(\widetilde{M}, +, \cdot, \leq_p\right)$ is a quasi module over R.

A_1 : Clearly $\left(\widetilde{M}, +\right)$ is a commutative semigroup with identity θ and '$\leq_p$' is a partial order in $\widetilde{M}$.

A_2 : Let $y \in \widetilde{M}$ and $r\,(\neq 0) \in R$. Then $\forall x \in M$, $x \leq_p \omega \Rightarrow x+y \leq_p \omega + y = \omega$ and $r\cdot x = rx \leq_p r\cdot\omega = \omega$. Also for any $x \in \widetilde{M}$, $x \leq_p x \Rightarrow x+y \leq_p x+y$ and $r\cdot x \leq_p r\cdot x$. Since $0\cdot y = \theta$, $\forall y \in \widetilde{M}$ so $0\cdot x \leq_p 0\cdot z$ whenever $x \leq_p z$ $\left(x, z \in \widetilde{M}\right)$.

A_3 : (i) For $r \neq 0$ and $x \in \widetilde{M}$ we have $r\cdot(x+\omega) = r\cdot\omega = \omega = r\cdot x + r\cdot\omega$ and $0\cdot(x+\omega) = \theta = 0\cdot x + 0\cdot\omega$

(ii) If $rr' \neq 0$ then $r\cdot(r'\cdot\omega) = \omega = (rr')\cdot\omega$, otherwise $r\cdot(r'\cdot\omega) = \theta = (rr')\cdot\omega$

(iii) If $r + r' = 0$ but not both 0 then $(r+r')\cdot\omega = \theta \leq_p \omega = r\cdot\omega + r'\cdot\omega$

(iv) $1_R \cdot x = x$, $\forall x \in \widetilde{M}$, 1_R being the multiplicative identity of R.

(v) $0\cdot x = \theta$, $\forall x \in \widetilde{M}$

The remaining cases follow immediately from the fact that M is a module over R.

A_4 : Here $\left[\widetilde{M}\right]_0 = M$. Since $\omega + (-1_R)\cdot\omega = \omega \neq \theta$ and $m + (-1_R)\cdot m = m - m = \theta$, $\forall m \in M$ we have $x + (-1_R)\cdot x = \theta$ iff $x \in M = \left[\widetilde{M}\right]_0$.

A_5 : For each $x \in M$, $x \leq_p x$ and for ω we have $m \leq_p \omega$, $\forall m \in M$

Thus it follows that $\left(\widetilde{M}, +, \cdot, \leq_p\right)$ is a quasi module over R, where M is the set of all one order elements of $\widetilde{M}$.

This example shows that every module is contained in a quasi module.

In this example if we consider $M = \mathbb{C}$, the vector space of all complex numbers as a module over the unitary ring $\mathbb{Z}$ then the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ becomes a quasi module over $\mathbb{Z}$, provided we define $0.\infty = 0$ and $z < \infty$, $\forall z \in \mathbb{C}$.

Example 2.4. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$. Then under the usual addition, $\mathbb{Z}^+$ is a commutative semigroup with the identity 0. Also it is a partially ordered set with respect to the usual order ($\leq$) of integers. If we define the ring multiplication '$\cdot$' : $\mathbb{Z} \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by $(m, n) \longmapsto |m|n$, then it is a routine work to verify that $(\mathbb{Z}^+, +, \cdot, \leq)$ is a quasi module over $\mathbb{Z}$. Here the set of all one order elements is given by $[\mathbb{Z}^+]_0 = \{0\}$.

Example 2.5. Let $\mathbb{Z}^+[x]$ be the set of all polynomials with coefficients taken from $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$. Then with respect to the usual addition $(+)$ of polynomials it is a commutative semigroup with the identity, viz. 'zero polynomial' $O(x)$. Let us define the ring multiplication '$\cdot$' : $\mathbb{Z} \times \mathbb{Z}^+[x] \longrightarrow \mathbb{Z}^+[x]$ by $(m, a_0 + a_1x + \cdots + a_nx^n) \longmapsto |m|(a_0 + a_1x + \cdots + a_nx^n)$. Again if $f(x), g(x) \in \mathbb{Z}^+[x]$, where $f(x) := a_0 + a_1x + \cdots + a_nx^n$ ($a_n \neq 0$) and $g(x) := b_0 + b_1x + \cdots + b_mx^m$ ($b_m \neq 0$) then we define $f(x) \preccurlyeq g(x)$ if $\deg f(x) \leq \deg g(x)$ and $a_i \leq b_i$, $\forall\, i = 0, 1, \ldots, n\,(= \deg f(x))$. Then clearly '$\preccurlyeq$' is a partial order in $\mathbb{Z}^+[x]$. It now follows that $(\mathbb{Z}^+[x], +, \cdot, \preccurlyeq)$ is a quasi module over the unitary ring $\mathbb{Z}$; the set of all one order elements of $\mathbb{Z}^+[x]$ is given by $\left[\mathbb{Z}^+[x]\right]_0 = \left\{O(x)\right\}$.

Example 2.6. Let $\mathbb{Q}^+ := \left\{\frac{p}{q} : p, q \in \mathbb{Z}^+, q \neq 0, \gcd(p,q) = 1\right\}$ i.e. the set of all non-negative rational numbers. Then with respect to the usual addition of rationals, $\mathbb{Q}^+$ becomes a commutative semigroup with zero (0) as the identity element. We define the ring multiplication '$\odot$' by elements of the ring $\mathbb{Z}$ by, $(r, \frac{p}{q}) \longmapsto |r|\frac{p}{q}$ $(r \in \mathbb{Z})$. The partial order '$\leq$' on $\mathbb{Q}^+$ is defined as $\frac{p_1}{q_1} \leq \frac{p_2}{q_2} \Leftrightarrow p_1 \leq p_2$ and $q_1 = q_2$. We now show that under these operations and partial order $(\mathbb{Q}^+, +, \odot, \leq)$ becomes a qmod over the unitary ring $\mathbb{Z}$. First of all, it is clear that '$\leq$' is truly a partial order on $\mathbb{Q}^+$. We are only to show the following to establish this example of qmod.

$\mathbf{A_2}$: If $x, y \in \mathbb{Q}^+$ with $x \leq y$ then for any $z \in \mathbb{Q}^+$ we have $z + x \leq z + y$ and for any $r \in \mathbb{Z}$ we have $|r|x \leq |r|y$ i.e. $r \odot x \leq r \odot y$.

$\mathbf{A_3}$: (i) $r \odot (x + y) = |r|(x + y) = |r|x + |r|y = r \odot x + r \odot y$, $\forall\, r \in \mathbb{Z}$, $\forall\, x, y \in \mathbb{Q}^+$.

(ii) $r_1 \odot (r_2 \odot x) = |r_1||r_2|x = |r_1r_2|x = (r_1r_2) \odot x$, $\forall\, r_1, r_2 \in \mathbb{Z}$, $\forall\, x \in \mathbb{Q}^+$.

(iii) $(r_1 + r_2) \odot x = |r_1 + r_2|x \leq |r_1|x + |r_2|x = r_1 \odot x + r_2 \odot x$, $\forall\, r_1, r_2 \in \mathbb{Z}$, $\forall\, x \in \mathbb{Q}^+$.

(iv) $1 \odot x = x$, $\forall\, x \in \mathbb{Q}^+$.

(v) $0 \odot x = 0$, $\forall\, x \in \mathbb{Q}^+$ and $r \odot 0 = 0$, $\forall\, r \in \mathbb{Z}$.

$\mathbf{A_4}$: $[\mathbb{Q}^+]_0 := \left\{\frac{p}{q} \in \mathbb{Q}^+ : \frac{r}{s} \not\leq \frac{p}{q}, \forall\, \frac{r}{s} \in \mathbb{Q}^+ \smallsetminus \{\frac{p}{q}\}\right\} = \{0\}$. Again, $1 \odot \frac{p}{q} + (-1) \odot \frac{p}{q} = 0 \Leftrightarrow \frac{p}{q} + \frac{p}{q} = 0 \Leftrightarrow 2\frac{p}{q} = 0 \Leftrightarrow \frac{p}{q} = 0$.

$\mathbf{A_5}$: For each $\frac{p}{q} \in \mathbb{Q}^+$ we have $0 \leq \frac{p}{q}$.

Thus $(\mathbb{Q}^+, +, \odot, \leq)$ is a qmod over $\mathbb{Z}$.

Example 2.7. Let $\{p_1, p_2, \ldots\}$ be a complete enumeration of all primes in order i.e. $2 = p_1 < p_2 < \cdots$. Now any integer $m \geq 1$ can be expressed uniquely as $m = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_i^{\alpha_i}\ldots$, where $\alpha_i \in \mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$, $\forall\, i$ and all but finitely many α_i's are zero. Thus we can identify the integer m with the sequence $(\alpha_1, \alpha_2, \ldots)$ in $\mathbb{Z}^+$. In other words, we can say that m can be identified with an element of $(\mathbb{Z}^+)^{\mathbb{N}}$ whose all but finitely many terms are zero and for convenience let us denote this set as $(\mathbb{Z}^+)^{\mathbb{N}}_{00}$ i.e.

$$(\mathbb{Z}^+)^{\mathbb{N}}_{00} := \Big\{(\alpha_1, \alpha_2, \dots) : \alpha_i \in \mathbb{Z}^+, \alpha_i = 0 \text{ for all but finitely many } i\text{'s}\Big\}$$

Let us first introduce some notations : if $\alpha := (\alpha_1, \alpha_2, \dots) \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$ and $p := (p_1, p_2, \dots)$ be the sequence of *all* primes in strictly increasing order, as stated above, we denote $p^\alpha := p_1^{\alpha_1} p_2^{\alpha_2} \dots$ which is valid since all but finitely many factors in this infinite product are 1. Also if $\alpha, \beta \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$ then $p^\alpha p^\beta = p^{\alpha+\beta}$, where $\alpha+\beta$ is the sequence in $(\mathbb{Z}^+)^{\mathbb{N}}_{00}$ obtained by term by term addition of α and β. Again if r is any non-negative integer then $(p^\alpha)^r = p^{r\alpha}$. Thus the usual product of two integers (≥ 1) can be viewed as the sum of two elements in $(\mathbb{Z}^+)^{\mathbb{N}}_{00}$; also for any integer $m\,(\geq 1)$ and $r \in \mathbb{Z}^+$ the exponent operation m^r can be viewed as the operation $r\alpha := (r\alpha_1, r\alpha_2, \dots)$, where $m := p^\alpha$ and $\alpha := (\alpha_1, \alpha_2, \dots) \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$. These facts now culminate into the following example of quasi module.

$(\mathbb{Z}^+)^{\mathbb{N}}_{00}$ is a commutative semigroup with respect to the usual term by term addition of two sequences; it contains an identity element, namely zero sequence '0' (i.e. the sequence all of whose terms are zero). With the help of the unitary ring $\mathbb{Z}$, we define a ring multiplication '$\cdot$': $\mathbb{Z} \times (\mathbb{Z}^+)^{\mathbb{N}}_{00} \longrightarrow (\mathbb{Z}^+)^{\mathbb{N}}_{00}$ as $(r, \alpha) \longmapsto |r|\alpha$. We now define an order '$\preccurlyeq$' by $\alpha \preccurlyeq \beta$ iff $p^\alpha \leq p^\beta$. It is obvious that $\preccurlyeq$ is a partial order in $(\mathbb{Z}^+)^{\mathbb{N}}_{00}$. We show below that $(\mathbb{Z}^+)^{\mathbb{N}}_{00}$ is a qmod over $\mathbb{Z}$ with respect to the aforesaid operations and partial order.

$\mathbf{A_2}$: Let $\alpha, \beta, \gamma \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$ with $\alpha \preccurlyeq \beta$. Then $p^\alpha \leq p^\beta \Rightarrow p^\gamma p^\alpha \leq p^\gamma p^\beta \Rightarrow p^{\gamma+\alpha} \leq p^{\gamma+\beta}$ $\Rightarrow \gamma+\alpha \preccurlyeq \gamma+\beta$. Again for any $r \in \mathbb{Z}$ we have $(p^\alpha)^{|r|} \leq (p^\beta)^{|r|} \Rightarrow p^{|r|\alpha} \leq p^{|r|\beta} \Rightarrow r\cdot\alpha \preccurlyeq r\cdot\beta$.

$\mathbf{A_3}$: Let $\alpha, \beta \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$ and $n, n_1, n_2 \in \mathbb{Z}$. Also let $\alpha := \big(\alpha_i\big)_{i\in\mathbb{N}}$, $\beta := \big(\beta_i\big)_{i\in\mathbb{N}}$. Then

(i) $n \cdot (\alpha + \beta) = \Big(|n|(\alpha_i + \beta_i)\Big)_{i\in\mathbb{N}} = \Big(|n|\alpha_i\Big)_{i\in\mathbb{N}} + \Big(|n|\beta_i\Big)_{i\in\mathbb{N}} = n \cdot \alpha + n \cdot \beta$.

(ii) $n_1 \cdot (n_2 \cdot \alpha) = n_1 \cdot \Big(|n_2|\alpha_i\Big)_{i\in\mathbb{N}} = \Big(|n_1||n_2|\alpha_i\Big)_{i\in\mathbb{N}} = \Big(|n_1 n_2|\alpha_i\Big)_{i\in\mathbb{N}} = (n_1 n_2) \cdot \alpha$.

(iii) $(n_1 + n_2) \cdot \alpha = \Big(|n_1 + n_2|\alpha_i\Big)_{i\in\mathbb{N}}$. Now $|n_1 + n_2|\alpha_i \leq |n_1|\alpha_i + |n_2|\alpha_i$, $\forall i \in \mathbb{N}$. So $p_i^{|n_1+n_2|\alpha_i} \leq p_i^{|n_1|\alpha_i} p_i^{|n_2|\alpha_i}$, $\forall i \in \mathbb{N} \Rightarrow p^{|n_1+n_2|\alpha} \leq p^{|n_1|\alpha+|n_2|\alpha} \Rightarrow (n_1 + n_2)\cdot\alpha \preccurlyeq n_1\cdot\alpha + n_2\cdot\alpha$.

(iv) $1 \cdot \alpha = (|1|\alpha_i)_{i\in\mathbb{N}} = \alpha$.

(v) $0 \cdot \alpha = (|0|\alpha_i)_{i\in\mathbb{N}} = 0$ and $n \cdot 0 = 0$.

$\mathbf{A_4}$: Since any $\alpha \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$ corresponds to an integer ≥ 1 it follows that, zero sequence '0' is the only one order element of $(\mathbb{Z}^+)^{\mathbb{N}}_{00}$. Again $1\cdot\alpha + (-1)\cdot\alpha = 0 \Leftrightarrow \Big(\alpha_i + \alpha_i\Big)_{i\in\mathbb{N}} = 0$ $\Leftrightarrow 2\alpha_i = 0, \forall i \Leftrightarrow \alpha = 0$. So axiom A_4 follows.

$\mathbf{A_5}$: For each $\alpha \in (\mathbb{Z}^+)^{\mathbb{N}}_{00}$, since $p^\alpha \geq 1 = p^0$ we have $0 \preccurlyeq \alpha$.

Thus it follows that $\Big((\mathbb{Z}^+)^{\mathbb{N}}_{00}, +, \cdot, \preccurlyeq\Big)$ is a quasi module over $\mathbb{Z}$.

3 Order morphism

In this section we introduce a morphism-like structure between two quasi modules over a common unitary ring and study some of its properties.

Definition 3.1. A mapping $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) is called an *order-morphism* if

(i) $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in X$

(ii) $f(rx) = rf(x)$, $\forall\, r \in R$, $\forall\, x \in X$

(iii) $x \leq y$ $(x, y \in X) \Rightarrow f(x) \leq f(y)$

(iv) $p \leq q$ $\big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$, where $\uparrow A := \{x \in X : x \geq a$ for some $a \in A\}$ and $\downarrow A := \{x \in X : x \leq a$ for some $a \in A\}$ for any $A \subseteq X$.

A surjective (injective, bijective) order-morphism is called an *order-epimorphism* (*order-monomorphism*, *order-isomorphism*).

Note 3.2. If $f : X \longrightarrow Y$ be an order-morphism and θ, θ' be the identity elements of X, Y respectively then $f(\theta) = f(0.\theta) = 0.f(\theta) = \theta'$. Again if x_0 be an one order element of X then so is $f(x_0)$ of Y. In fact, $x_0 \in X_0 \Rightarrow x_0 - x_0 = \theta \Rightarrow f(x_0) - f(x_0) = \theta'$ $\Rightarrow f(x_0) \in Y_0$. Also if $y_0 \in Y_0 \cap f(X)$ then $\exists\, x \in f^{-1}(y_0) \Rightarrow \exists\, x_0 \in X_0$ such that $x_0 \leq x \Rightarrow f(x_0) \leq f(x) = y_0 \Rightarrow f(x_0) = y_0$ [$\because y_0$ is an one order element of Y]. Thus $f^{-1}(y_0) \cap X_0 \neq \emptyset$.

Before proceeding further let us first introduce the following concept which will be useful in the sequel.

Definition 3.3. A subset Y of a qmod X is said to be a *sub quasi module* (*subqmod* in short) if Y itself be a quasi module with all the compositions of X being restricted to Y.

Note 3.4. A subset Y of a qmod X (over a unitary ring R) is a sub quasi module iff Y satisfies the following conditions :

(i) $rx + sy \in Y$, $\forall\, r, s \in R$, $\forall\, x, y \in Y$.

(ii) $Y_0 \subseteq X_0 \cap Y$, where $Y_0 := \big\{z \in Y : y \not\leq z, \forall\, y \in Y \smallsetminus \{z\}\big\}$

(iii) $\forall\, y \in Y$, $\exists\, y_0 \in Y_0$ such that $y_0 \leq y$

If Y be a subqmod of X then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$ we have $X_0 \cap Y \subseteq Y_0$.

Proposition 3.5. *If $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) be an order-morphism then $f(M) := \{f(m) : m \in M\}$ is a subqmod of Y, for any subqmod M of X.*

Proof. For $x, y \in M$ and $r, s \in R$ we have $rf(x) + sf(y) = f(rx + sy) \in f(M)$, since $rx+sy \in M$ for, M is a subqmod of X. Clearly, $f(M) \cap Y_0 \subseteq [f(M)]_0$. Now let $y \in [f(M)]_0$

$\Rightarrow \exists\, m \in M$ such that $y = f(m)$. So $\exists\, p \in M_0$ such that $p \leq m \Rightarrow f(p) \leq f(m) \Rightarrow f(p) = f(m) = y$ [$\because y$ is an one order element of $f(M)$ and $f(p) \in f(M)$]. Since $M_0 = M \cap X_0$ so $p \in X_0 \Rightarrow f(p)$ is an one order element of Y and hence $y = f(p) \in Y_0 \cap f(M)$. Thus $[f(M)]_0 \subseteq f(M) \cap Y_0$. Therefore $[f(M)]_0 = f(M) \cap Y_0$. Again for any $m \in M$, $\exists\, m_0 \in M_0$ such that $m_0 \leq m \Rightarrow f(m_0) \leq f(m)$. Here $f(m_0)$ is an one order element of Y and hence $f(m_0) \in [f(M)]_0$. Thus it follows that $f(M)$ is a subqmod of Y. □

Proposition 3.6. *Let X, Y, Z be three qmods over an unitary ring R and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two order-morphisms. Then their composition $g \circ f : X \longrightarrow Z$ is an order-morphism, provided f is onto.*

Proof. $(g \circ f)(rx_1 + x_2) = g(f(rx_1 + x_2)) = g(rf(x_1) + f(x_2)) = r.(g \circ f)(x_1) + (g \circ f)(x_2)$, $\forall\, x_1, x_2 \in X$ and $\forall\, r \in R$. Moreover, $x_1 \leq x_2\, (x_1, x_2 \in X) \Rightarrow f(x_1) \leq f(x_2) \Rightarrow g(f(x_1)) \leq g(f(x_2)) \Rightarrow (g \circ f)(x_1) \leq (g \circ f)(x_2)$.

Now let $z_1, z_2 \in (g \circ f)(X)$ such that $z_1 \leq z_2$. Let $x \in (g \circ f)^{-1}(z_1)$. Then $(g \circ f)(x) = z_1 \Rightarrow g(f(x)) = z_1 \Rightarrow f(x) \in g^{-1}(z_1) \subseteq\downarrow g^{-1}(z_2) \Rightarrow f(x) \leq y$, for some $y \in g^{-1}(z_2)$ $\Rightarrow g(y) = z_2$. Now f being onto, $y \in f(X)$ and hence $x \in\downarrow f^{-1}(y) \Rightarrow x \leq x'$, where $f(x') = y$. Therefore $(g \circ f)(x') = g(f(x')) = g(y) = z_2 \Rightarrow x \in\downarrow (g \circ f)^{-1}(z_2)$.
$\therefore (g \circ f)^{-1}(z_1) \subseteq\downarrow (g \circ f)^{-1}(z_2)$.
Again $x_0 \in (g \circ f)^{-1}(z_2) \Rightarrow (g \circ f)(x_0) = z_2 \Rightarrow g(f(x_0)) = z_2 \Rightarrow f(x_0) \in g^{-1}(z_2) \subseteq\uparrow g^{-1}(z_1)$ $\Rightarrow f(x_0) \geq y'$, for some $y' \in g^{-1}(z_1)$. So $g(y') = z_1$. Now f being onto, $y' \in f(X)$ and hence $x_0 \in\uparrow f^{-1}(y') \Rightarrow x_0 \geq x''$, where $f(x'') = y'$. Therefore $(g \circ f)(x'') = g(f(x'')) = g(y') = z_1$ $\Rightarrow x_0 \in\uparrow (g \circ f)^{-1}(z_1)$.
$\therefore (g \circ f)^{-1}(z_2) \subseteq\uparrow (g \circ f)^{-1}(z_1)$.
Thus it follows that $(g \circ f)$ is an order-morphism. □

Proposition 3.7. *If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ (X, Y, Z being qmods over the same unitary ring R) be two order-morphisms such that $g \circ f : X \longrightarrow Z$ is also an order-morphism then*

(i) *$g \circ f$ is onto iff both f, g are onto;*

(ii) *$g \circ f$ is injective iff both f, g are injective.*

Proof. First of all, $g \circ f$ is an order-morphism provided f is onto. So (i) is immediate. For (ii), we are only to show that g is injective whenever $g \circ f$ is injective. Actually $g \circ f$ is injective implies g is injective on $f(X)$; in fact, if $g(f(x_1)) = g(f(x_2))$ for $f(x_1) \neq f(x_2)$ (and hence for $x_1 \neq x_2$) then injectivity of $g \circ f$ would be contradicted. Since for $g \circ f$ to be an order-morphism f needs to be onto, (ii) follows. □

Remark 3.8. The above proposition readily implies that two order-isomorphisms, after composition, generates again an order-isomorphism; inverse of an order-isomorphism is an order-isomorphism and the identity map on any qmod is an order-isomorphism. Thus 'order-isomorphism' induces an equivalence relation on the collection of all qmods over the same unitary ring and we can identify two such qmods related by this equivalence relation.

Definition 3.9. Let $f : X \longrightarrow Y$ (X, Y being two qmods over the same unitary ring R) be an order-morphism. We define $\ker f := \Big\{(x, y) \in X \times X : f(x) = f(y)\Big\}$ and call it the '*kernel of f*'.

It is immediate from definition that $(x, x) \in \ker f$, $\forall x \in X$ and thus if we write $\Delta := \Big\{(x, x) : x \in X\Big\}$ then $\Delta \subseteq \ker f$, equality holds iff f is injective.

We now show that $\ker f$ is a subqmod of $X \times X$, but for doing so we have to first discuss the Cartesian product of qmods.

4 Arbitrary product of quasi modules

In this section we shall discuss arbitrary product of quasi modules and show that the product is also a quasi module.

Definition 4.1. Let $\{X_\mu : \mu \in \Lambda\}$ be an arbitrary family of quasi modules over the unitary ring R. Let $X := \prod\limits_{\mu \in \Lambda} X_\mu$ be the Cartesian product of these quasi modules defined as : $x \in X$ if and only if $x : \Lambda \longrightarrow \bigcup\limits_{\mu \in \Lambda} X_\mu$ is a map such that $x(\mu) \in X_\mu$, $\forall\, \mu \in \Lambda$. Then by the axiom of choice we know that X is nonempty, since Λ is nonempty and each X_μ contains at least the additive identity θ_μ (say).
Let us denote $x_\mu := x(\mu)$, $\forall\, \mu \in \Lambda$. Also we write each $x \in X$ as $x = (x_\mu)$, where $x_\mu = p_\mu(x)$, $p_\mu : X \longrightarrow X_\mu$ being the projection map, $\forall\, \mu \in \Lambda$. Now we define addition, ring multiplication and partial order as follows : for $x = (x_\mu), y = (y_\mu) \in X$ and $r \in R$
(i) $x + y = (x_\mu + y_\mu)$; (ii) $r.x = (rx_\mu)$; (iii) $x \leq y$ if $x_\mu \leq y_\mu$, $\forall\, \mu \in \Lambda$.

We now show that $(X, +, ., \leq)$ is a quasi module over R.
A_1 : Clearly X is a commutative semigroup with identity θ, where $\theta = (\theta_\mu)$.
A_2 : $x \leq y \Rightarrow x_\mu \leq y_\mu$, $\forall \mu \in \Lambda \Rightarrow x_\mu + z_\mu \leq y_\mu + z_\mu$ and $rx_\mu \leq ry_\mu$, $\forall \mu \in \Lambda$ and $\forall\, r \in R$ $\Rightarrow x + z \leq y + z$ and $rx \leq ry$, where $z = (z_\mu) \in X$.
A_3 : For $x = (x_\mu), y = (y_\mu) \in X$ and for $r, s \in R$ we have
(i) $r(x + y) = \Big(r(x_\mu + y_\mu)\Big) = \Big(rx_\mu + ry_\mu\Big) = (rx_\mu) + (ry_\mu) = rx + ry$.
(ii) $r(sx) = r(sx_\mu) = rs(x_\mu) = rsx$.

(iii) each X_μ being a quasi module, $(r+s)x_\mu \le rx_\mu + sx_\mu$, $\forall \mu \in \Lambda \Rightarrow (r+s)x \le rx + sx$.

(iv) $1.x = (1.x_\mu) = (x_\mu) = x$, 1 being the multiplicative identity of R.

(v) $0.x = (0.x_\mu) = (\theta_\mu) = \theta$. Again $r.\theta = (r.\theta_\mu) = (\theta_\mu) = \theta$.

$A_4 : x - x = \theta \Leftrightarrow x_\mu - x_\mu = \theta_\mu, \forall \mu \in \Lambda \Leftrightarrow x_\mu \in [X_\mu]_0, \forall \mu \in \Lambda$, where $[X_\mu]_0$ is the set of all one order elements of X_μ. We claim that $X_0 = \Big\{(x_\mu) \in X : x_\mu \in [X_\mu]_0, \forall \mu \in \Lambda\Big\} \equiv \prod\limits_{\mu\in\Lambda} [X_\mu]_0$. In fact, $x = (x_\mu) \notin \prod\limits_{\mu\in\Lambda}[X_\mu]_0 \Rightarrow x_\lambda \notin [X_\lambda]_0$ for some $\lambda \in \Lambda$. Then $\exists\, i_\lambda \in [X_\lambda]_0$ such that $i_\lambda \le x_\lambda$, $i_\lambda \ne x_\lambda$. Let $y = (y_\mu)$ where $y_\mu = x_\mu$, $\mu \ne \lambda$ and $y_\lambda = i_\lambda$. Then $y \le x$, $y \ne x \Rightarrow x \notin X_0$. Conversely if $x_\mu \in [X_\mu]_0$, $\forall \mu \in \Lambda$ then $x = (x_\mu) \in X_0$. Thus $X_0 = \prod\limits_{\mu\in\Lambda}[X_\mu]_0$. So $x - x = \theta \Leftrightarrow x \in X_0$.

A_5 : Let $x = (x_\mu) \in X$. Then $x_\mu \in X_\mu$, $\forall \mu \in \Lambda \Rightarrow \exists\, t_\mu \in [X_\mu]_0$, $\forall \mu \in \Lambda$ such that $t_\mu \le x_\mu$, $\forall \mu \in \Lambda \Rightarrow t = (t_\mu) \le (x_\mu) = x$, where $t \in X_0$.

$\therefore (X, +, ., \le)$ is a quasi module over R.

Proposition 4.2. *Let $\{X_i : i \in \Lambda\}$ be an arbitrary family of quasi modules over an unitary ring R and $X := \prod\limits_{i\in\Lambda} X_i$ be the product qmod of these qmods. Then each projection map $p_j : X \longrightarrow X_j$ is an order-epimorphism.*

Proof. Let $x = (x_i), y = (y_i) \in X$ and $r \in R$. Then $p_j(rx + y) = rx_j + y_j = rp_j(x) + p_j(y)$. Again if $x \le y$ then $x_i \le y_i$, $\forall\, i \in \Lambda \Rightarrow p_j(x) \le p_j(y)$.

Now let $a, b \in p_j(X) = X_j$ (since every projection map is onto) with $a \le b$. Let $x = (x_i) \in p_j^{-1}(a)$. Then $p_j(x) = a$ i.e. $x_j = a$. We choose $y = (y_i)$ where $y_i = x_i$ for $i \ne j$ and $y_j = b$. Then $x \le y$ and $p_j(y) = y_j = b$ i.e. $y \in p_j^{-1}(b)$. Thus we have $p_j^{-1}(a) \subseteq\downarrow p_j^{-1}(b)$. Similarly we can show that $p_j^{-1}(b) \subseteq\uparrow p_j^{-1}(a)$. So p_j being onto the proposition follows. □

Proposition 4.3. *Let $f : X \longrightarrow Y$ be an order-morphism (X, Y being two qmods over an unitary ring R). Then $\ker f$ is a subqmod of $X \times X$.*

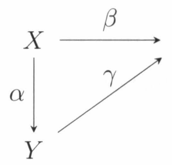
Proof. For $(x_1, y_1), (x_2, y_2) \in \ker f$ and $r, s \in R$ we have $f(rx_1 + sx_2) = rf(x_1) + sf(x_2) = rf(y_1) + sf(y_2) = f(ry_1 + sy_2) \Rightarrow (rx_1 + sx_2, ry_1 + sy_2) \in \ker f$ i.e. $r(x_1, y_1) + s(x_2, y_2) \in \ker f$.

Now let $(x, y) \in \ker f$ but $(x, y) \notin X_0 \times X_0$. Without loss of generality assume that $x \notin X_0$. Then $\exists\, a \in X_0$ such that $a \le x$. So $f(a) \le f(x) = f(y)$. Now f being an order-morphism, $\exists\, z \in f^{-1}(f(a))$ such that $z \le y$. Then $\exists\, t \in X_0$ such that $t \le z \Rightarrow f(t) \le f(z) = f(a)$. Now a being one order, $f(a)$ is so and hence $f(t) = f(a) \Rightarrow (a, t) \in \ker f$. Also $(a, t) \le (x, y)$ and $(a, t) \ne (x, y)$. This ensures that $(x, y) \notin [\ker f]_0$. Thus we have $[\ker f]_0 \subseteq \ker f \cap (X_0 \times X_0)$. Also from this argument we find that for any $(x, y) \in \ker f$, $\exists\, (a, t) \in \ker f \cap (X_0 \times X_0) = [\ker f]_0$ such that $(a, t) \le (x, y)$ $\big($if (x, y) itself be of order one we need not find $(a, t)\big)$. The proposition then follows from the note 3.4. □

5 Order isomorphism theorem

In this section we present an isomorphism theorem between quasi modules.

Lemma 5.1. *Let X, Y, Z be three quasi modules over the unitary ring R, $\alpha : X \longrightarrow Y$ be an order-epimorphism and $\beta : X \longrightarrow Z$ be an order-morphism such that* $\ker\alpha \subseteq \ker\beta$. *Then $\exists$ a unique order-morphism $\gamma : Y \longrightarrow Z$ such that $\gamma \circ \alpha = \beta$.*



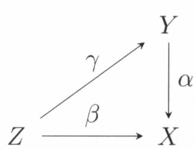
Proof. We first show that if an order-morphism $\gamma : Y \longrightarrow Z$ exists satisfying $\gamma\circ\alpha = \beta$ then that must be unique. In fact, if γ' be another such order-morphism then $\gamma \circ \alpha = \beta = \gamma' \circ \alpha$. This shows that γ, γ' coincide on $\alpha(X)$. α being onto, γ, γ' really coincide on Y.

To prove the existence let $y \in Y$. α being order-epimorphism, $\alpha^{-1}(y) \neq \emptyset$. Now $\ker\alpha \subseteq \ker\beta \Rightarrow \beta$ is constant on $\alpha^{-1}(y)$. So it is reasonable to define $\gamma(y) := \beta(\alpha^{-1}(y))$, $\forall\, y \in Y$. Clearly then $\gamma \circ \alpha = \beta$. Now let $y, y' \in Y$, $r \in R$ and $x \in \alpha^{-1}(y)$, $x' \in \alpha^{-1}(y')$. Then $\gamma(y) = \beta(x)$ and $\gamma(y') = \beta(x')$. Now α being an order-morphism, we have $y + ry' = \alpha(x) + r\alpha(x') = \alpha(x + rx') \Rightarrow x + rx' \in \alpha^{-1}(y + ry')$. Then β being an order-morphism we have $\gamma(y) + r\gamma(y') = \beta(x) + r\beta(x') = \beta(x + rx') = \gamma(y + ry')$.

Next let $y \leq y'$ $(y, y' \in Y)$. Then $\alpha^{-1}(y) \subseteq\downarrow \alpha^{-1}(y')$. So for $x \in \alpha^{-1}(y)$, $\exists\, x' \in \alpha^{-1}(y')$ such that $x \leq x'$. Thus $\gamma(y) = \beta(x) \leq \beta(x') = \gamma(y')$.

For declaring γ to be an order-morphism it now remains to show that $\gamma^{-1}(z) \subseteq\downarrow \gamma^{-1}(z')$ and $\gamma^{-1}(z') \subseteq\uparrow \gamma^{-1}(z)$, whenever $z \leq z' \left[z, z' \in \gamma(Y)\right]$. To prove the first inclusion let $y \in \gamma^{-1}(z)$. Then $\alpha^{-1}(y) \subseteq \beta^{-1}(z) \subseteq\downarrow \beta^{-1}(z')$. So for $x \in \alpha^{-1}(y)$, $\exists\, x' \in \beta^{-1}(z')$ such that $x \leq x'$. Then $y = \alpha(x) \leq \alpha(x') = y'$ (say). Now $\gamma(y') = \gamma(\alpha(x')) = \beta(x') = z'$ $\Rightarrow y' \in \gamma^{-1}(z')$, where $y \leq y'$. The second inclusion can be similarly disposed of. □

Lemma 5.2. *Let X, Y, Z be three quasi modules over the unitary ring R, $\alpha : Y \longrightarrow X$ be an order-monomorphism and $\beta : Z \longrightarrow X$ be an order-morphism such that $\alpha(Y) = \beta(Z)$. Then $\exists$ a unique order-epimorphism $\gamma : Z \longrightarrow Y$ such that the following diagram commutes.*



Proof. α being an order-monomorphism it follows that α is an order-isomorphism from Y onto the sub quasi module $\alpha(Y)\ \big(\equiv \beta(Z)\big)$ of X. Thus we may define $\gamma := \alpha^{-1} \circ \beta$. It then follows from the remark 3.8 and proposition 3.6 that γ is an order-morphism, since β is 'onto' the domain of α^{-1}. Since $\alpha^{-1}(\beta(Z)) = Y$, so γ is surjective. Also $\alpha \circ \gamma = \beta$ i.e. the above diagram is commutative.

If, together with γ, any other γ' makes the above diagram commutative then, $\alpha \circ \gamma = \beta = \alpha \circ \gamma' \Longrightarrow \gamma = \gamma'$ (since α is an order-isomorphism). □

Before going to prove the isomorphism theorem we need to construct a quotient structure which again necessitates the introduction of the concept of 'congruence'. So let us define this concept first.

Definition 5.3. An equivalence relation E, defined on a quasi module X over an unitary ring R, is said to be a *congruence* on X if,
(i) $(x, y) \in E \Rightarrow (a + x, a + y) \in E,\ \forall\, a \in X$
(ii) $(x, y) \in E \Rightarrow (rx, ry) \in E,\ \forall\, r \in R$
(iii) $x \leq y \leq z$ and $(x, z) \in E \Rightarrow (x, y) \in E\ \big($and hence $(y, z) \in E\big)$
(iv) $a \leq x \leq b$ and $(x, y) \in E \Rightarrow \exists\, c, d \in X$ with $c \leq y \leq d$ such that $(a, c), (b, d) \in E$.

Proposition 5.4. *If $\phi : X \longrightarrow Y\ (X, Y$ being two qmods over an unitary ring $R)$ be an order-morphism then* $\ker \phi$ *is a congruence on X.*

Proof. Clearly $\ker \phi$ is an equivalence relation on X. Let $(x, y) \in \ker \phi$, $a \in X$ and $r \in R$. Then $\phi(a+x) = \phi(a)+\phi(x) = \phi(a)+\phi(y) = \phi(a+y)$ and $\phi(rx) = r\phi(x) = r\phi(y) = \phi(ry)$. Thus $(a + x, a + y), (rx, ry) \in \ker \phi$. Again $x \leq y \leq z \Rightarrow \phi(x) \leq \phi(y) \leq \phi(z)$. So $(x, z) \in \ker \phi \Rightarrow \phi(x) = \phi(z) = \phi(y) \Rightarrow (x, y) \in \ker \phi$.

Next let $b \geq x$. Then $\phi(b) \geq \phi(x) = \phi(y) \Rightarrow y \in \phi^{-1}(\phi(y)) \subseteq\downarrow \phi^{-1}(\phi(b)) \Rightarrow \exists\, d \in \phi^{-1}(\phi(b))$ such that $y \leq d$. Now $\phi(d) = \phi(b) \Rightarrow (b, d) \in \ker \phi$. Similarly for $a \leq x$ we have $\phi(a) \leq \phi(x) = \phi(y)$. So $y \in \phi^{-1}(\phi(y)) \subseteq\uparrow \phi^{-1}(\phi(a)) \Rightarrow \exists\, c \in \phi^{-1}(\phi(a))$ such that $y \geq c$. Now $\phi(c) = \phi(a) \Rightarrow (a, c) \in \ker \phi$. Thus $\ker \phi$ is a congruence on X. □

We now give a quotient structure on X using the above congruence. For this let us construct the quotient set $X/\ker \phi := \big\{[x] : x \in X\big\}$, where $[x]$ is the equivalence class containing x obtained by the congruence $\ker \phi$. We define addition, ring multiplication and partial order on $X/\ker \phi$ as follows : For $x, y \in X$ and $r \in R$,
(i) $[x] + [y] := [x + y]$; (ii) $r[x] := [rx]$; (iii) $[x] \leq [y]$ if and only if $\phi(x) \leq \phi(y)$.

Theorem 5.5. *If $\phi : X \longrightarrow Y\ (X, Y$ being two qmods over an unitary ring $R)$ be an order-morphism then $X/\ker \phi$ is a quasi module over R.*

Proof. A_1 : Clearly $(X/\ker\phi, +)$ is a commutative semigroup with identity $[\theta]$, where θ is the identity of X.

A_2 : Let $[x], [y], [z] \in X/\ker\phi$ and $[x] \leq [y]$. Now, $\phi(x+z) = \phi(x) + \phi(z) \leq \phi(y) + \phi(z) = \phi(y+z) \Rightarrow [x+z] \leq [y+z] \Rightarrow [x] + [z] \leq [y] + [z]$. Also, $\phi(rx) = r\phi(x) \leq r\phi(y) = \phi(ry)$, $\forall\, r \in R \Rightarrow [rx] \leq [ry] \Rightarrow r[x] \leq r[y]$, $\forall\, r \in R$.

A_3 : (i) $r([x] + [y]) = r[x+y] = [rx + ry] = [rx] + [ry] = r[x] + r[y]$

(ii) $r(s[x]) = r[sx] = [rsx] = rs[x]$, where $r, s \in R$

(iii) $(r+s)x \leq rx + sx \Rightarrow \phi((r+s)x) \leq \phi(rx) + \phi(sx) \Rightarrow (r+s)[x] = [(r+s)x] \leq [rx] + [sx] = r[x] + s[x]$

(iv) $1_R[x] = [x]$

(v) $0[x] = [0x] = [\theta]$ and $r[\theta] = [r\theta] = [\theta]$, $\forall\, r \in R$

A_4 : $[x] + (-1)[x] = [\theta] \Leftrightarrow [x] + [-x] = [\theta] \Leftrightarrow [x - x] = [\theta] \Leftrightarrow \phi(x - x) = \phi(\theta) \Leftrightarrow \phi(x) - \phi(x) = \theta'$ (where θ' is the identity in Y) $\Leftrightarrow \phi(x) \in Y_0$. Now the set of all one order elements of $X/\ker\phi$ is given by $\left[X/\ker\phi\right]_0 := \left\{[x] \in X/\ker\phi : [y] \nleq [x], \forall\, [y] \neq [x]\right\} = \left\{[x] : \phi(y) \nleq \phi(x), \forall\, \phi(y) \neq \phi(x)\right\} = \left\{[x] : \phi(x) \in [\phi(X)]_0 = \phi(X) \cap Y_0\right\}$ [$\because \phi(X)$ is a subqmod of Y]. Thus we have $[x] + (-1)[x] = [\theta]$ if and only if $[x] \in \left[X/\ker\phi\right]_0$.

A_5 : Let $[x] \in X/\ker\phi$. Then $\exists\, p \in X_0$ such that $p \leq x \Rightarrow \phi(p) \leq \phi(x) \Rightarrow [p] \leq [x]$. Here p being an one order element of X, $\phi(p)$ is so in Y and hence $[p] \in \left[X/\ker\phi\right]_0$. □

Proposition 5.6. *Let $\phi : X \longrightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism. Then the canonical map $\pi : X \longrightarrow X/\ker\phi$ defined by $\pi(x) := [x]$, $\forall\, x \in X$ is an order-epimorphism.*

Proof. Since ϕ is an order-morphism it follows immediately that π satisfies the first three axioms of an order-morphism. Also π is an onto map. So we are only to show that $\pi^{-1}\big([x]\big) \subseteq\downarrow \pi^{-1}\big([y]\big)$ and $\pi^{-1}\big([y]\big) \subseteq\uparrow \pi^{-1}\big([x]\big)$, whenever $[x] \leq [y]$ in $X/\ker\phi$. For this let $a \in \pi^{-1}\big([x]\big)$. Then $[a] = \pi(a) = [x] \Rightarrow \phi(a) = \phi(x) \leq \phi(y)$. Now $a \in \phi^{-1}(\phi(a)) \subseteq\downarrow \phi^{-1}(\phi(y)) \Rightarrow \exists\, b \in \phi^{-1}(\phi(y))$ such that $a \leq b$. Again $\phi(b) = \phi(y) \Rightarrow \pi(b) = [b] = [y]$. Thus we have $a \in\downarrow \pi^{-1}\big([y]\big)$ i.e. $\pi^{-1}\big([x]\big) \subseteq\downarrow \pi^{-1}\big([y]\big)$, whenever $[x] \leq [y]$ in $X/\ker\phi$. Similarly $\pi^{-1}\big([y]\big) \subseteq\uparrow \pi^{-1}\big([x]\big)$. □

We now have the following isomorphism theorem.

Theorem 5.7. *If $\phi : X \longrightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism then $X/\ker\phi$ is order-isomorphic to $\phi(X)$.*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \phi(X) \subseteq Y \\ {\scriptstyle \pi}\downarrow & \nearrow_{\psi} & \\ X/\ker\phi & & \end{array}$$

Proof. $\ker\pi := \{(x,y) : \pi(x) = \pi(y)\} = \{(x,y) : [x] = [y]\} = \{(x,y) : \phi(x) = \phi(y)\} = \ker\phi$. Since $\phi : X \longrightarrow \phi(X) \subseteq Y$ is an order-morphism and $\pi : X \longrightarrow X/\ker\phi$ is an order-epimorphism (by proposition 5.6), by lemma 5.1 we can find a unique order-morphism $\psi : X/\ker\phi \longrightarrow \phi(X)$ such that $\psi \circ \pi = \phi$. Now $\phi : X \longrightarrow \phi(X)$ is onto implies ψ is onto. Again $\psi[x] = \psi[y] \Rightarrow \psi(\pi(x)) = \psi(\pi(y)) \Rightarrow \phi(x) = \phi(y) \Rightarrow (x,y) \in \ker\phi$ $\Rightarrow [x] = [y]$, where $[x],[y] \in X/\ker\phi$. Therefore ψ is injective and hence bijective. □

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