Rev. Acad. Canar. Cienc., XVI (Núms. 1-2), 9-14 (2004) (publicado en julio de 2005)

SEPARATION OF ZEROS OF PARA-ORTHOGONAL RATIONAL FUNCTIONS

A. Bultheel*, P. González-Vera†, E. Hendriksen‡ & Olav Njåstad§

Abstract

We generalize a result by L. Golinskii [6] on separation of the zeros of para-orthogonal polynomials on the unit circle to a similar result for para-orthogonal rational functions.

Resumen

En este trabajo se extiende un resultado de Golinskii [6] sobre separación de ceros de polinomios para-ortogonales sobre la circunferencia unidad al caso de funciones racionales para-ortoganales.

1 Introduction

Every probability measure on the unit circle gives rise to an orthonormal sequence $\{\rho_n\}_{n=0}^{\infty}$ of polynomials, so called Szegő polynomials. See for example [7, 8]. Invariant para-orthogonal polynomials are polynomials of the form $c_n[\rho_n(z) + \tau\rho_n^*(z)]$, $|\tau| = 1$, $c_n \neq 0$, where $\rho_n^*(z) = z^n\overline{\rho_n(1/\overline{z})}$. These polynomials have all their zeros on the unit circle, and they are all simple. The zeros are nodes in a quadrature formula with positive weights which is exact on the space $\text{span}\{1/z^{n-1}, \ldots, 1, \ldots, z^{n-1}\}$. See e.g. [7]. An equivalent representation of the invariant para-orthogonal polynomials is as the class of all polynomials of the form $d_n[\rho_n^*(z)\overline{\rho_n^*(w)} - \rho_n(z)\overline{\rho_n(w)}]$, $|w| = 1$, $d_n \neq 0$. For a given w, the value $z = w$ is a zero of this polynomial. It was shown by Golinskii [6] that the zeros of two consecutive of these polynomials (for a given w) separate each other when the zero $z = w$ is not included among the zeros of the polynomial of highest degree.

The aim of this note is to prove a similar result for orthogonal rational functions on the unit circle. In sections 2 and 3 we give a brief summary of relevant basic properties of such functions. For a more comprehensive treatment, see [3]. In section 4 we give a proof of the indicated result, in the main following the reasoning of Golinskii.

By a quite different approach, Cantero, Moral and Velázquez [5] obtained separation results that contain the result of Golinskii.

---

*Department of Computer Science, K.U.Leuven, Belgium. The work of this author is partially supported by the Fund for Scientific Research (FWO), projects "CORFU: Constructive study of orthogonal functions", grant #G.0184.02 and the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the author.

†Department Análisis Math., Univ. La Laguna, Tenerife, Spain. The work of this author was partially supported by the scientific research project PB96-1029 of the Spanish D.G.E.S.

‡Department of Mathematics, University of Amsterdam, The Netherlands.

§Department of Math. Sc., Norwegian Univ. of Science and Technology, Trondheim, Norway

2 Orthogonal rational functions

In the following, $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ denotes the unit circle and $\mathbb{E}$ the exterior of the closed unit disk.

Let a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of not necessarily distinct points in $\mathbb{D}$ be given. We define

$$\zeta_0 = 1, \quad \zeta_n(z) = z_n \frac{z-\alpha_n}{1-\overline{\alpha}_n z}, \quad n = 1, 2, \ldots, \tag{2.1}$$

where $z_n = -|\alpha_n|/\alpha_n$ if $\alpha_n \neq 0$ and $z_n = 1$ if $\alpha_n = 0$. Furthermore we define the Blaschke products B_n by

$$B_0 = 1, \quad B_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots. \tag{2.2}$$

The functions $\{B_0, B_1, \ldots, B_n\}$ span the space $\mathcal{L}_n$ consisting of all functions of the form $f(z) = P(z)/\pi(z)$, where $P \in \mathcal{P}_n$ (the space of polynomials of degree at most n) and

$$\pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z). \tag{2.3}$$

In general we define for any function $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the superstar transform f^* by $f^*(z) = B_n(z) f_*(z)$, where $f_*(z) = \overline{f(1/\overline{z})}$. Note that f^* also belongs to $\mathcal{L}_n$.

Let μ be a probability measure on $\mathbb{T}$, with associated inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_{\mathbb{T}} f(t)\overline{g(t)} d\mu(t). \tag{2.4}$$

We shall use the notation ϕ_n for the elements of the orthonormal basis for $\mathcal{L}_n$ which is ordered such that $\phi_0 = 1$ and $\phi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ for $k = 1, 2, \ldots, n$. We may then write $\phi_n(z) = p_n(z)/\pi_n(z)$, $\phi_n^*(z) = q_n(z)/\pi_n(z)$ where $p_n \in \mathcal{P}_n$, $q_n \in \mathcal{P}_n$.

We note that if $\alpha_n = 0$ for all n, then $B_n(z) = z^n$, $\mathcal{L}_n = \mathcal{P}_n$ and ϕ_n, ϕ_n^* are orthonormal polynomials with respect to μ and their reciprocals. For motivations for studying the rational generalizations of orthogonal polynomials introduced above, we refer to [3, 4].

Let $k_n(z, w)$ denote the reproducing kernel for $\mathcal{L}_n$, i.e.,

$$k_n(z, w) = \sum_{j=0}^{n} \phi_j(z)\overline{\phi_j(w)}. \tag{2.5}$$

The orthonormal functions ϕ_n satisfy

$$\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)} = [1 - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}]k_n(z, w), \tag{2.6}$$

$$\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)} = [1 - \zeta_n(z)\overline{\zeta_n(w)}]k_n(z, w). \tag{2.7}$$

It follows easily from these formulas that

$$\begin{array}{ll} |\phi_n(z)| < |\phi_n^*(z)| & \text{for } z \in \mathbb{D} \\ |\phi_n(z)| = |\phi_n^*(z)| & \text{for } z \in \mathbb{T} \\ |\phi_n(z)| > |\phi_n^*(z)| & \text{for } z \in \mathbb{E}. \end{array} \tag{2.8}$$

(Note that $|\zeta_n(z)| < 1$ for $z \in \mathbb{D}$, $|\zeta_n(z)| = 1$ for $z \in \mathbb{T}$ and $|\zeta_n(z)| > 1$ for $z \in \mathbb{E}$.) Furthermore all the zeros of ϕ_n lie in $\mathbb{D}$. Simple examples (e.g. with μ the normalized Lebesgue measure and $\alpha_n = 0$ for all n, which gives $\phi_n(z) = z^n$) show that the zeros may be multiple.

For more exhaustive treatments, see e.g., [1, 3].

3 Para-orthogonal rational functions

Quadrature formulas with positive weights and nodes on $\mathbb{T}$ have important uses. Of special interest are such formulas which integrate exactly all functions in spaces of the form $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_q, g_* \in \mathcal{L}_p\}$ with as large value of $p+q$ as possible. The zeros of ϕ_n can not be used as nodes, since they lie in $\mathbb{D}$ (and may even be multiple). It turns out that so-called invariant para-orthogonal functions give rise to such quadrature formulas, exact on $\mathcal{L}_{n-1,n-1}$ (while no quadrature formula as specified can be exact on $\mathcal{L}_{n-1,n}$ of on $\mathcal{L}_{n,n-1}$). See [2, 3].

A function Q_n in $\mathcal{L}_n$ is called invariant if $Q_n^*(z) = k_n Q_n(z)$ for some $k_n \neq 0$. It is called para-orthogonal if $\langle Q_n, f\rangle = 0$ for all $f \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, where $\mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}$, while $\langle Q_n, 1\rangle \neq 0$ and $\langle Q_n, B_n\rangle \neq 0$. (These concepts are direct generalizations of corresponding concepts in the polynomial case, ie., when $\alpha_n = 0$ for all n. These were introduced and studied in [7].)

It can be shown that the invariant para-orthogonal rational functions are exactly functions of the form $c_n Q_n(z,\tau)$, $c_n \neq 0$, where

$$Q_n(z,\tau) = [\phi_n(z) + \tau\phi_n^*(z)], \quad \tau \in \mathbb{T}. \tag{3.1}$$

Furthermore, $Q_n(z,\tau)$ has exactly n simple zeros, all of them lying on $\mathbb{T}$. See [3].

Now consider a function $d_n\Omega_n(z,w)$, $d_n \neq 0$, where

$$\Omega_n(z,w) = [\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}]. \tag{3.2}$$

We may write

$$\Omega_n(z,w) = -\overline{\phi_n(w)}[\phi_n(z) + (-\overline{\left[\frac{\phi_n^*(w)}{\phi_n(w)}\right]}\phi_n^*(z)]. \tag{3.3}$$

Because of (2.8) we have for $w \in \mathbb{T}$ that $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} \in \mathbb{T}$. Thus $\Omega_n(z,w)$ is a function of the form $c_n Q_n(z,\tau)$ as in (3.1). On the other hand, for each $\tau \in \mathbb{T}$, there are n values of w in $\mathbb{T}$ such that $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} = \tau$. (Note that for a given τ, $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} = \tau$ may be written as an algebraic equation of degree n in w, and that according to (2.8), $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} \in \mathbb{T}$ if and only if $w \in \mathbb{T}$. See also [3, Thm. 5.2.1].) Thus the class of functions of the form $c_n Q_n(z,\tau)$, $c_n \neq 0$, $\tau \in \mathbb{T}$ as given in (3.1) is exactly the same as the class of functions $d_n\Omega_n(z,w)$, $d_n \neq 0$, $w \in \mathbb{T}$ as given in (3.2).

4 Separation of zeros

In [6] Golinskii showed that in the polynomial case, i.e., when all α_n equal zero, a certain separation property of the zeros of two consecutive polynomials $\Omega_n(z,w)$ (for fixed w) holds. We shall prove a similar result in the general rational case. The result as well as the proof are rather straightforward generalizations of Golinskii's discussion in the polynomial case.

In the following, w denotes a fixed point on $\mathbb{T}$. We observe that $[1 - \zeta_n(z)\overline{\zeta_n(w)}] = 0$ if and only if $z = w$. It then follows from (2.5)-(2.6) and (3.2) that $z = w$ is a zero of $\Omega_n(z,w)$ for all n, and that the remaining zeros of $\Omega_n(z,w)$ are exactly the zeros of $k_{n-1}(z,w)$.

Now assume that z_0 is a common zero of $\Omega_n(z,w)$ and $\Omega_{n+1}(z,w)$, $z_0 \neq w$. Note that z_0 has to be on $\mathbb{T}$. It follows from (2.6) and the definition (3.2) that $k_n(z_0,w) = 0$ and

$k_{n-1}(z_0, w) = 0$, hence also $\phi_n(z_0)\overline{\phi_n(w)} = 0$. This is impossible since all the zeros of ϕ_n lie in $\mathbb{D}$. Consequently $\Omega_n(z,w)$ and $\Omega_{n+1}(z,w)$ have no common zeros except $z = w$.

Now for each n let $z_{n,k} = e^{i\theta_{n,k}}$, $k = 0, 1, \dots, n-1$, be the zeros of $\Omega_n(z,w)$, with $z_{n,0} = w$, ordered such that

$$\theta_{n,0} < \theta_{n,1} < \cdots < \theta_{n,n-1} < \theta_{n,0} + 2\pi. \tag{4.1}$$

Theorem 4.1 *The zeros of $\Omega_n(z,w)$ included $z = w$ and the zeros of $\Omega_{n+1}(z,w)$ not included $z = w$ separate each other in the sense that*

$$\theta_{n,0} < \theta_{n+1,1} < \theta_{n,1} < \theta_{n+1,2} < \cdots < \theta_{n,n-1} < \theta_{n+1,n}. \tag{4.2}$$

Proof. Consider the function

$$\Gamma_n(z) = \Gamma_n(z,w) = \frac{k_n(z,w)}{\Omega_n(z,w)}. \tag{4.3}$$

It follows from the foregoing discussion that the zeros of $k_n(z,w)$ are exactly the points $z_{n+1,1}, \dots, z_{n+1,n}$ while the zeros of $\Omega_n(z,w)$ are the points $z_{n,0}, \dots, z_{n,n-1}$. Thus $\Gamma_n(z)$ has simple zeros at the points $z_{n+1,1}, \dots, z_{n+1,n}$ and simple poles at the points $z_{n,0}, \dots, z_{n,n-1}$. (Recall that $\Omega_n(z,w)$ and $\Omega_{n+1}(z,w)$ have no common zeros except $z = w$. Also note that the terms $\pi_n(z)$ in the numerator and the denominator cancel.) Expressing $k_n(z,w)$ by (2.7) we may write

$$\Gamma_n(z) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{[1 - \zeta_n(z)\overline{\zeta_n(w)}][\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}]}. \tag{4.4}$$

We introduce the function b_n defined by

$$b_n(z) = \frac{\phi_n(z)}{\phi_n^*(z)}. \tag{4.5}$$

We note that b_n is holomorphic in $\mathbb{D} \cup \mathbb{T}$ and maps $\mathbb{D}$ onto $\mathbb{D}$, $\mathbb{T}$ onto $\mathbb{T}$, according to (2.8). In terms of this function, $\Gamma_n(z)$ may be written as

$$\Gamma_n(z) = \frac{1 - \zeta_n(z)\overline{\zeta_n(w)}b_n(z)\overline{b_n(w)}}{[1 - \zeta_n(z)\overline{\zeta_n(w)}][1 - b_n(z)\overline{b_n(w)}]} \tag{4.6}$$

and hence by a simple calculation

$$\Gamma_n(z) = \frac{1}{2}\left[\frac{1 + b_n(z)\overline{b_n(w)}}{1 - b_n(z)\overline{b_n(w)}} + \frac{1 + \zeta_n(z)\overline{\zeta_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}}\right]. \tag{4.7}$$

The Möbius transformation $z \to \frac{1+z}{1-z}$ maps $\mathbb{D}$ onto the open right half-plane $\mathbb{H}$ and $\mathbb{T}$ onto the extended imaginary axis $\hat{\mathbb{I}}$. Taking into account the mapping properties of the function b_n stated above, we find that each of the two terms in (4.7) maps $\mathbb{D}$ onto $\mathbb{H}$ and $\mathbb{T}$ onto $\hat{\mathbb{I}}$. The function $\Gamma_n(z)$ then has the same property. In other words, $\Gamma_n(z)$ is a lossless Carathéodory function.

A rational lossless Carathéodory function has the property that the zeros and poles separate each other. For the sake of completeness, we sketch the proof.

The function $\Gamma_n(z)$ may be written in the from

$$\Gamma_n(z) = ic + \sum_{k=0}^{n-1} \lambda_k \frac{z + z_{n,k}}{z - z_{n,k}}, \tag{4.8}$$

where $\lambda_k > 0$ and c is a real constant. (See e.g. [7].) We find that

$$\lambda_k \frac{e^{i\theta} + e^{i\theta_{n,k}}}{e^{i\theta} - e^{i\theta_{n,k}}} = -2i\lambda_k \frac{\sin(\theta - \theta_{n,k})}{|e^{i\theta} - e^{i\theta_{n,k}}|^2}.$$

It follows that

$$\text{Im}\left(\lambda_k \frac{e^{i\theta} + e^{i\theta_{n,k}}}{e^{i\theta} - e^{i\theta_{n,k}}}\right) > 0 \quad \text{for } \theta < \theta_{n,k}, \quad \text{Im}\left(\lambda_k \frac{e^{i\theta} + e^{i\theta_{n,k}}}{e^{i\theta} - e^{i\theta_{n,k}}}\right) < 0 \quad \text{for } \theta > \theta_{n,k}.$$

When $\theta \to \theta_{n,k}$, the term $\lambda_k \frac{z+z_{n,k}}{z-z_{n,k}} = \lambda_k \frac{e^{i\theta}+e^{i\theta_{n,k}}}{e^{i\theta}-e^{i\theta_{n,k}}}$ in (4.8) dominates, hence we may conclude that

$$\lim_{\theta \to \theta_{n,k}^-} \frac{1}{i}\Gamma_n(e^{i\theta}) = +\infty, \qquad \lim_{\theta \to \theta_{n,k}^+} \frac{1}{i}\Gamma_n(e^{i\theta}) = -\infty.$$

Thus the image of the arc $\{z = e^{i\theta} : \theta_{n,k} < \theta < \theta_{n,k+1}\}$ by the mapping $z \to \Gamma_n(z)$ is the whole imaginary axis $\mathbb{I}$. Consequently (at least) one of the zeros of $\Gamma_n(z)$ must lie on this arc. Taking into account the ordering for general n indicated in (4.1) and the fact that $\Gamma_n(z)$ has the same number of zeros and poles, we conclude that (4.2) holds.

This completes the proof of the theorem. □

References

[1] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. The computation of orthogonal rational functions and their interpolating properties. *Numer. Algorithms*, 2(1):85–114, 1992.

[2] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and quadrature on the unit circle. *Numer. Algorithms*, 3:105–116, 1992.

[3] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonality rational functions on the unit circle: from the scalar to the matrix case. Technical Report TW401, Department of Computer Science, K.U.Leuven, July 2004.

[5] M.J. Cantero, L. Moral, and L. Velázquez. Measures and para-orthogonal polynomials on the unit circle. *East J. Approx.*, 8:447–464, 2002.

[6] L. Golinskii. Quadrature formula and zeros of para-orthogonal polynomials on the unit circle. *Acta Math. Hungar.*, 96:169–186, 2002.

[7] W.B. Jones, O. Njåstad, and W.J. Thron. Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle. *Bull. London Math. Soc.*, 21:113–152, 1989.

[8] G. Szegő. *Orthogonal polynomials*, volume 33 of *Amer. Math. Soc. Colloq. Publ.* Amer. Math. Soc., Providence, Rhode Island, 4th edition, 1975. First edition 1939.