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ON THE GENERALIZED HANKEL TYPE INTEGRAL TRANSFORMATION OF GENERALIZED FUNCTIONS-II

S.P.MALGONDE
Deptt.of Maths.,
College of Engineering,
Kopargaon-423 603
Dist:Ahmednager,(M.S.), INDIA.

ABSTRACT

In this paper the generalized Hankel type integral transformation depending on three real parameters defined by

$$F_2(y)=(F_{2,\mu,\alpha,\beta,\nu}f)(y)=\nu\beta\int_0^{\infty} x^{-1-2\alpha+2\nu}(xy)^{\alpha}J_{\mu}[\beta(xy)^{\nu}]f(x)dx \quad (\mu\geq -1/2)$$

where $J_{\mu}(x)$ is the Bessel function of the first kind of order μ,which reduces to almost all the Hankel,generalized Hankel and Hankel type integral transformations,is extended to certain spaces of generalized functions by the kernel method in such a way that the theory of E.L. Koh and A.H.Zemanian in relation with the Hankel transformation

$$F(y) = (h_{\mu}f)(y) = \int_0^{\infty} \sqrt{(xy)}J_{\mu}(xy)f(x)dx \qquad (\mu\geq -1/2)$$

appears then as a particular case for $\nu=1,\beta=1,\alpha=1/2$. An inversion theorem is established by interpreting convergence in the weak distributional sense.The theory thus developed is applied to solve certain initial value problems.

KEY WORDS Genralized Hankel type transformations,generalized functions, countable union spaces,inversion theorem,adjoint method, operational calculus,generalized Cauchy problems.

1 INTRODUCTION

Some generalizations of the classical Hankel transformation [15 & 17]

$$F(y) = (h_\mu f)(y) = \int_0^\infty \sqrt{(xy)}\, J_\mu(xy) f(x)\,dx \qquad (\mu \ge -1/2) \qquad (1.1)$$

were given by many authors from time to time. Recently Malgonde [4] introduced the generalized Hankel type transformation depending on three real parameters (α, β, ν) defined by

$$F_1(y) = (F_{1,\mu,\alpha,\beta,\nu} f)(y) = \nu\beta\, y^{-1-2\alpha+2\nu} \int_0^\infty (xy)^\alpha J_\mu[\beta (xy)^\nu] f(x)\,dx \quad (\mu \ge -1/2) \quad (1.2)$$

where α, β and ν are any arbitrary real numbers and $J_\mu(z)$ is the Bessel function of the first kind of order μ. Later on Malgonde [5] extended (1.2) to certain spaces of generalized functions by kernel method. The F_1-transformation (1.2) was also extended by Malgonde and Bandewar [8] to certain generalized functions of slow growth through a generalization of mixed Parseval equation (1.7) given below.

A new variant of the generalized Hankel type integral transformation depending on three real parameter (α, β, ν) is defined by

$$F_2(y) = (F_{2,\mu,\alpha,\beta,\nu} f)(y) = \nu\beta \int_0^\infty x^{-1-2\alpha+2\nu} (xy)^\alpha J_\mu[\beta (xy)^\nu] f(x)\,dx \quad (\mu \ge -1/2) \quad (1.3)$$

where α, β and ν are any arbitrary real numbers and $J_\mu(z)$ is the Bessel function of the first kind of order μ.

Recently two variants of Hankel type integral transformations defined by

$$F(y) = (F_{1,\mu,\nu} f)(y) = y^{1+2\mu} \int_0^\infty (xy)^{-\mu} J_\nu(xy) f(x)\,dx \qquad (1.4)$$

$$F(y) = (F_{2,\mu,\nu} f)(y) = \int_0^\infty x^{1+2\mu} (xy)^{-\mu} J_\nu(xy) f(x)\,dx \qquad (1.5)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν $(\nu \ge -1/2)$ and μ is an arbitrary real parameter, which are particular cases of (1.2) and (1.3) respectively for $\beta=1, \nu=1, \alpha=-\mu$, $\mu=\nu$, have been extended to certain space of generalized functions by Malgonde [6 and 7].

In view of the general nature of the kernel involved in the transformations (1.2) and (1.3) on specializing the parameters we obtain almost all the Hankel transformations [14,15,16,17], the Hankel-Schwartz transform [12], the Hankel-Clifford transform [9], amongst others.

We shall use quite a few times the asymptotic expansions

$$J_\mu[\beta x^\nu] \cong \sqrt{\frac{2\nu}{\beta\pi}}\ x^{-\nu/2}\cos[\beta x^\nu-(\pi/2)(\mu+1/2)] + O(\beta x^\nu)^{-3/2}, \text{ as } x\to\infty$$

$$J_\mu[\beta x^\nu] \cong [\Gamma(1+\nu)]^{-1}(\beta x^\nu/2)^\mu \quad , \text{ as } x\to 0 \quad ,\mu\geq -1/2 \qquad (1.6)$$

and the following some of the important classical results [4].

THEOREM 1 (Inversion formula)

If f(x) is of bounded variation into a neighbourhood of the point $x=x_0>0$, $\mu\geq -1/2$ and the integral $\int_0^\infty |f(x)|x^{\alpha-\nu/2}dx$ exists, then

$$\lim_{R\to\infty} \nu\beta\int_0^R y^{-1-2\alpha+2\nu}(x_0 y)^\alpha J_\mu[\beta(x_0 y)^\nu]F_2(y)dydx=\frac{1}{2}[f(x_0+0)+f(x_0-0)].$$

THEOREM 2 (Mixed Parseval's equation)

If $f(x)x^{\alpha+\mu}$ and $F_2(y)y^{\mu\nu-\alpha-1+2\nu}$ are in $L_1(0,\infty)$, $F_1(y)=F_{1,\mu,\alpha,\beta,\nu}$ [f(x)](y) and $F_2(y)=F_{2,\mu,\alpha,\beta,\nu}[g(x)](y)$ then for $\mu \geq -1/2$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_1(y)F_2(y)dy \qquad (1.7)$$

According to Méndez [10] the equality (1.7) is called the mixed Parseval's equation for the F_2-transformation or $F_{2,\mu,\alpha,\beta,\nu}$-transformation

As it is well-known there exist two ways to define an integral transform of generalized functions, the adjoint and the kernel method. The adjoint method has been employed by Zemanian [17], Méndez [10], Lee [3], Schuitman [11], amongst others. The kernel method was used by Koh and Zemanian [2], Dube and Pandey [1], amongst others.

In the present paper we extend the F_2-transformation (1.3) to other spaces of generalized functions following a different procedure called the kernel method. Theorems on smoothness , boundedness, inversion and uniqueness , together with an operation -transform formula for a

Bessel-type differential operator are presented.

The notation and terminology used here are those of Zemanian [17]. Throughout this work I denotes the open interval $(0,\infty)$. D(I) denotes the space of smooth functions whose supports are compact subsets of I. We assign to D(I) the topology that makes its dual D'(I) the space of Schwartz's distribution on I [13].E(I) and E'(I) are, respectively, the space of smooth functions on I and the space of distributions with compact supports on I.We use the following operators:

$$D=D_x=\frac{d}{dx}\ ,\ (\Delta_{\alpha,\mu,\nu})^k=(x^{-\alpha-\mu\nu}D_x\, x^{2\mu\nu+1}D_x\, x^{-\mu\nu+\alpha+1-2\nu})^k \text{ and}$$

$$(\Delta^*_{\alpha,\mu,\nu})^k=(x^{-\mu\nu+\alpha+1-2\nu}D_x\, x^{2\mu\nu+1}D_x\, x^{-\alpha-\mu\nu})^k \quad \text{for } k=0,1,2,\ldots .$$

Let α,μ and ν be any arbitrary real numbers.$H_{1,\alpha,\mu,\nu}$ and $H_{2,\alpha,\mu,\nu}$ denote the linear spaces consisting of all smooth complex-valued functions $\phi(x)$ on I such that, for every pair of non-negative integers (m,k), the numbers

$$\gamma^{1,\alpha,\mu,\nu}_{m,k}(\phi(x)) = \sup_{0<x<\infty} |x^m (x^{1-2\nu}D)^k x^{-\mu\nu+\alpha+1-2\nu}\phi(x)| \qquad (1.8)$$

and

$$\gamma^{2,\alpha,\mu,\nu}_{m,k}(\phi(x)) = \sup_{0<x<\infty} |x^m (x^{1-2\nu}D)^k x^{-\alpha-\mu\nu}\,\phi(x)| \qquad (1.9)$$

exist respectively. The set of seminorms $\{\gamma^{i,\alpha,\mu,\nu}_{m,k}\}^{\infty}_{m,k=0}$ where i=1,2 generates the topology of $H_{1,\alpha,\mu,\nu}$ and $H_{2,\alpha,\mu,\nu}$ respectively.The duals of $H_{1,\alpha,\mu,\nu}$ and $H_{2,\alpha,\mu,\nu}$ are denoted by $H'_{1,\alpha,\mu,\nu}$ and $H'_{2,\alpha,\mu,\nu}$ respectively

2.THE TESTING FUNCTION SPACES $\mathbb{H}_{\alpha,\mu,\nu,a}$ AND $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$ AND THEIR DUALS

Let a denote a positive real number and α μ,ν be any arbitrary real parameters.Then for each a,α,μ,and ν we define $\mathbb{H}_{\alpha,\mu,\nu,a}$ as the space of testing functions $\phi(x)$ defined on $0<x<\infty$ and for which

$$\eta_k^{\alpha,\mu,\nu,a}(\phi) = \sup_{0<x<\infty} |e^{-ax} x^{-\mu\nu+\alpha+1-2\nu} \Delta^k_{\alpha,\mu,\nu}\phi(x)| < \infty \qquad (2.1)$$

for k = 0,1,2,-----.

We assign to $\mathbb{H}_{\alpha,\mu,\nu,a}$ the topology generated by the countable multinorm $\{\eta_k^{\alpha,\mu,\nu,a}\}_{k=0}^{\infty}$. $\mathbb{H}_{\alpha,\mu,\nu,a}$ is Hausdorff space, since $\eta_0^{\alpha,\mu,\nu,a}$ is a norm on $\mathbb{H}_{\alpha,\mu,\nu,a}$. Moreover, $\mathbb{H}_{\alpha,\mu,\nu,a}$ is a locally convex linear space that satisfies the first axiom of countability. The dual space $\mathbb{H}'_{\alpha,\mu,\nu,a}$ consists of all continuous linear functionals on $\mathbb{H}_{\alpha,\mu,\nu,a}$.

Following Koh and Zemanian [2] and Malgonde [6], we now list some properties of these spaces.

(i) Let $\mu \geq -1/2$, a > 0 and α,ν be any arbitrary real parameters and $K_1(x,y) = \nu\beta x^{-1-2\alpha+2\nu}(xy)^{\alpha} J_{\mu}[\beta(xy)^{\nu}]$. For a fixed positive rea number y,

$$\frac{\partial^m}{\partial y^m}\Big[K_1(x,y)\Big] \in \mathbb{H}_{\alpha,\mu,\nu,a}, \quad m = 0,1,2,\dots .$$

(ii) $\mathbb{H}_{\alpha,\mu,\nu,a}$ is sequentially complete and therefore a Frechet space. Hence $\mathbb{H}'_{\alpha,\mu,\nu,a}$ is also sequentially complete.

(iii) If a>b>0, then $\mathbb{H}_{\alpha,\mu,\nu,b} \subset \mathbb{H}_{\alpha,\mu,\nu,a}$, and the topology of $\mathbb{H}_{\alpha,\mu,\nu,b}$ is stronger than that induced on it by $\mathbb{H}_{\alpha,\mu,\nu,a}$.

(iv) $H_{1\alpha,\mu,\nu}$ is a proper subset of $\mathbb{H}_{\alpha,\mu,\nu,a}$ for every choice of a>0, and the topology of $H_{1\alpha,\mu,\nu}$ is stronger than that induced on it by $\mathbb{H}_{\alpha,\mu,\nu,a}$.

(v) $D(I) \subset \mathbb{H}_{\alpha,\mu,\nu,a}$, and the topology of D(I) is stronger than that induced on it by $\mathbb{H}_{\alpha,\mu,\nu,a}$.

(vi) For every choice of α,μ,ν and a, $\mathbb{H}_{\alpha,\mu,\nu,a} \subset E(I)$. Moreover, it is dense in E(I) because $D(I) \subset \mathbb{H}_{\alpha,\mu,\nu,a}$ and D(I) is dense in E(I). The topology of $\mathbb{H}_{\alpha,\mu,\nu,a}$ is stronger than that induced on it by E(I). Hence, E'(I) can be identified with a subspace of $\mathbb{H}'_{\alpha,\mu,\nu,a}$.

(vii) The operation $\phi \to \Delta_{\alpha,\mu,\nu}\phi$ is a continuous linear mapping of $\mathbb{H}_{\alpha,\mu,\nu,a}$

into itself since

$$\eta_k^{\alpha,\mu,\nu,a}(\Delta_{\alpha,\mu,\nu}\phi) = \eta_{k+1}^{\alpha,\mu,\nu,a}(\phi) \quad \text{for } k = 0,1,2 \ldots .$$

(viii) Let f be an arbitrary element of $\mathbb{H}_{\alpha,\mu,\nu,a}$. Then there exist bounded measurable functions $g_i(x)$ defined for $x>0$ and $i=0,1,2\ldots,r$, where r is some nonnegative integer depending upon f, such that for an arbitrary $\phi \in D(I)$ we have

$$\langle f,\phi\rangle = \langle \sum_{i=0}^{r}(\Delta^*_{\alpha,\mu,\nu})^i \{e^{-ax} x^{-\mu\nu+\alpha+1-2\nu}(-D_x)\, g_i(x)\} , \phi(x)\rangle.$$

We turn now to the definition of a certain countable-union spaces $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$ that arise from the $\mathbb{H}_{\alpha,\mu,\nu,a}$ spaces. Our subsequent discussion takes on a simpler form when the $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$ spaces are used in place of the $\mathbb{H}_{\alpha,\mu,\nu,a}$ spaces.

Following Koh and Zemanian [2], $\mathbb{H}_{\alpha,\mu,\nu}(\sigma) = \bigcup_{p=1}^{\infty} \mathbb{H}_{\alpha,\mu,\nu,a_p}$ is the countable-union space, where $\{a_p\}_{p=1}^{\infty}$ is a monotonic sequence of positive numbers such that $a_p \to \sigma$ ($\sigma=+\infty$ is allowed). A generalized function f is $F'_{2,\mu,\alpha,\beta,\nu}$ -transformable if $f\in \mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ for some $\sigma>0$, where $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ is the dual of $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$.

In view of our definitions of $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$ and its dual, the following lemmas are immediate.

<u>LEMMA 1</u> For any fixed $y>0$, $\dfrac{\partial^m}{\partial y^m}[K_1(x,y)] \in \mathbb{H}_{\alpha,\mu,\nu}(\sigma)$, $m=0,1,2\ldots$ where $\sigma>0$.

<u>LEMMA 2</u> For every choice of $\sigma>0$, $H_{1\alpha,\mu,\nu} \subset \mathbb{H}_{\alpha,\mu,\nu}(\sigma)$,

and convergence in $H_{1,\alpha,\mu,\nu}$ implies convergence in $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$. The restriction of $f\in \mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ to $H_{1\alpha,\mu,\nu}$ is in $H'_{1\alpha,\mu,\nu}$ and convergence in $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ implies convergence in $H'_{1\alpha,\mu,\nu}$

<u>LEMMA 3</u> The operation $\phi\to \Delta_{\alpha,\mu,\nu}\phi$ is a continuous linear mapping of $\mathbb{H}_{\alpha,\mu,\nu}(\sigma)$ into itself. Hence the operation $f \to \Delta^*_{\alpha,\mu,\nu} f$ is a continuous linear mapping of $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ into itself [17].

As was indicated in note(vi), $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ contains all distributions of compact support on $I=(0,\infty)$. Similarly any conventional function f for some $a<\sigma$ is a member of $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$, as is every generalized derivative $(\Delta^{*}_{\alpha,\mu,\nu})^k f$, k=1,2,3. . .according to Lemma 3. Moreover, we may say that the members of $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$ are "generalized functions of exponential descent", since the multiform $\{\eta_k^{\alpha,\mu,\nu,a}\}$ shows that the testing functions $\phi\in\mathbb{H}_{\alpha,\mu,\nu,a}$ are at most of exponential growth.

3 THE GENERALIZED HANKEL TYPE INTEGRAL TRANSFORMATION $F'_{2,\mu,\alpha,\beta,\nu}$

Let $\mu\geq -1/2$ and α,ν be any arbitrary real numbers. In view of note (iii) Of § 2, to every $f\in\mathbb{H}'_{\alpha,\mu,\nu,a}$ there exists a unique real number σ_f (possibly, $\sigma_f=+\infty$) such that $f\in\mathbb{H}'_{\alpha,\mu,\nu,b}$ if $b<\sigma_f$ and $f\notin\mathbb{H}'_{\alpha,\mu,\nu,b}$ if $b>\sigma_f$. Therefore, $f\in\mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$. We define the μ^{th} order generalized Hankel type integral transform $F'_{2,\mu,\alpha,\beta,\nu}f$ of f as the application of f to the kernel $K_1(x,y)$; i.e.,

$$F_2(y) = (F'_{2,\mu,\alpha,\beta,\nu}f)(y) = \langle f(x)\ ,\ K_1(x,y)\rangle \qquad (3.1)$$

where $0<y<\infty$ and $\sigma_f>0$. The right hand side of (3.1) is meaningful by Lemma 1 for each $y>0$ and $\sigma_f>0$.

LEMMA 4 Let a and σ_f be fixed real numbers such that $0<a<\sigma_f$. For all fixed $y>0$, for $\mu\geq -1/2$ and for $0<x<\infty$

$$|e^{-ax}[\beta(xy)^{\nu}]^{-\mu}\ J_{\mu}[\beta(xy)^{\nu}]| < A_{\mu,\beta,\nu} \qquad (3.2)$$

where $A_{\mu,\beta,\nu}$ is a constant with respect to x and y.

PROOF: The proof is simple and can be verified as it was made by Koh and Zemanian [2].

THEOREM 3 (Analyticity of $F_2(y)$): For $y > 0$, let $F_2(y)$ be defined by (3.1). Then

$$\frac{d}{dy}F_2(y) = \langle f(x)\ ,\frac{\partial}{\partial y}K_1(x,y)\rangle.$$

PROOF: The proof can be easily verified following Koh & Zemanian [2].

THEOREM 4 (Boundedness of $F_2(y)$): Let $F_2(y)$ be defined by (3.1). Then $F_2(y)$ is bounded according to

$$|F_2(y)| \le \begin{cases} c\, y^{-1-\alpha+2\nu+\mu\nu} & \text{as } y \to 0+ \\ c\, y^{2\nu r-1-\alpha+2\nu+\mu\nu} & \text{as } y \to \infty \end{cases} \tag{3.3}$$

where c is a positive constant and r is non-negative integer.

PROOF: Proof is very similar to that of Koh and Zemanian [2].

In view of note (iv) of § 2 and Lemma 2, if f is in $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$ then f belongs to $H'_{1,\alpha,\mu,\nu}$ provided that $\mu \ge -1/2$. We now show that generalized Hankel type integral transform of $f \in \mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$ given by (3.1) is equal (in the sense of equality in $H'_{1,\alpha,\mu,\nu}$) to the generalized Hankel type integral transform of f as given in Malgonde and Bandewar [7 by

$$\langle F'_2 f, \phi \rangle = \langle f, F_1 \phi \rangle \tag{3.4}$$

for every $f \in H'_{1,\alpha,\mu,\nu}$ and $\phi \in H_{1,\alpha,\mu,\nu}$.

THEOREM 5 Let $f \in \mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$, $\phi \in \mathbb{H}_{1,\alpha,\mu,\nu}$, and $\mu \ge -1/2$. Then

$$\langle \langle f(x), K_1(x,y) \rangle, \phi(y) \rangle = \langle f(x) \int_0^\infty K_1(x,y)\phi(y)dy \rangle. \tag{3.5}$$

PROOF: Proof follows on the similar lines as that of Malgonde [5].

THEOREM 6 Let $F_2(y) = (F'_{2,\mu,\alpha,\beta,\nu} f)(y)$, $f \in \mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$ as in (3.1) where $y>0$. Let $\mu \ge -1/2$. Then, in the sense of convergence in $D'(I)$,

$$f(x) = \lim_{R\to\infty} \int_0^R F_2(y) K_2(x,y)dy \tag{3.6}$$

where $K_2(x,y) = \nu\beta\, y^{-1-2\alpha+2\nu} (xy)^\alpha J_\mu[\beta(xy)^\nu]$.

PROOF: Let $\phi(x) \in D(I)$. We wish to show that

$$\langle \int_0^R F_2(y) K_2(x,y)dy, \phi(x) \rangle \tag{3.7}$$

tends to $\langle f(x), \phi(x) \rangle$ as $R \to \infty$. From the smoothness of $F_2(y)$ and the fact that support of $\phi(x)$ is a compact subset of I, we may write

(3.7) as a repeated integral on (x,y) having a continuous integrand and a finite domain of integration. Hence we can change the order of integration and obtain

$$\int_0^{\infty}\phi(x)\int_0^R F_2(y)K_2(x,y)dydx=\int_0^R\langle f(t),K_1(t,y)\rangle\int_0^{\infty}\phi(x)K_2(x,y)dxdy. \qquad (3.8)$$

By an argument based on Riemann sums for the integral $\int_0^R .. dy$, the right side of (3.8) can be written as

$$\langle f(t) , \int_0^R K_1(t,y) \int_0^{\infty}\phi(x)K_2(x,y)dxdy \rangle \qquad (3.9)$$

By the formula in Malgonde [4]

$$\nu\beta\int_0^R y^{-1+2\nu} J_\mu[\beta(ty)^\nu]J_\mu[\beta(xy)^\nu]dy$$

$$=\frac{R^\nu}{x^{2\nu}-t^{2\nu}}[x^\nu J_{\mu+1}[\beta(xR)^\nu]J_\mu[\beta(tR)^\nu]-t^\nu J_{\mu+1}[\beta(tR)^\nu] J_\mu[\beta(xR)^\nu]$$

and the asymptotic representations of the Bessel functions enable us to show that for any a>0,the testing function in (3.9) converges in $\mathbb{H}_{\alpha,\mu,\nu,a}$ to $\phi(t)$ as $R\to\infty$. Since $f\in\mathbb{H}'_{\alpha,\mu,\nu,a}$ where $0<a<\sigma_f$,it follows that (3.9) converges to $\langle f(t),\phi(t)\rangle$ as $R\to\infty$. This proves the theorem.

THEOREM 7 (Uniqueness theorem):Let $F_2(y)=\mathcal{F}'_{2,\mu,\alpha,\beta,\nu}f(y)$ for y>0 and $G_2(y)=\mathcal{F}'_{2,\mu,\alpha,\beta,\nu}g(y)$ for y>0,f and g being in $\mathbb{H}'_{\alpha,\mu,\nu}(\sigma)$.
If $F_2(y)=G_2(y)$,for every y>0,then f=g in the sense of equality in D'(I).
PROOF :By Theorem 6 , $f-g = \lim_{R\to\infty}\int_0^R[F_2(y)-G_2(y)] K_2(x,y)dy = 0$.

4. AN OPERATION-TRANSFORM FORMULA

In this section,we shall apply the preceding theory in solving certain differential equations involving generalized functions.
We define the operator $\Delta^*_{\alpha,\mu,\nu}:\mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)\to\mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$ by the relation

$$\langle \Delta^*_{\alpha,\mu,\nu}f(x) , \phi(x) \rangle = \langle f(x) , \Delta_{\alpha,\mu,\nu}\phi(x)\rangle \qquad (4.1)$$

for all $f\in\mathbb{H}'_{\alpha,\mu,\nu}(\sigma_f)$ and $\phi\in\mathbb{H}_{\alpha,\mu,\nu}(\sigma_f)$, $\mu\geq -1/2$ and for α and ν arbitrary real numbers.It can be readily seen that