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Mittag-Leffler integral transform on $\mathcal{L}_{\nu,r}$-spaces

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Abstracts

The paper is devoted to the study of the integral transform

$$(\mathbf{E}_{\lambda,\sigma}f)(x) = \int_0^\infty E_{\lambda,\sigma}(-xt)f(t)dt \quad (x > 0)$$

with positive $\lambda > 0$ and complex σ involving the Mittag-Leffler function $E_{\lambda,\sigma}(z)$ in the kernel, on the space $\mathcal{L}_{\nu,r}$ of Lebesgue measurable functions f on $\mathbf{R}_+ = (0,\infty)$ such that

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty \ (1 \leq r < \infty), \quad ess\ sup_{t\in\mathbf{R}_+} [t^\nu |f(t)|] < \infty \ (r = \infty)$$

with real $\nu \in \mathbf{R}$, coinciding with the space $L^r(\mathbf{R}_+)$ $(1 \leq r \leq \infty)$ when $\nu = 1/r$. It is proved that the transform $\mathbf{E}_{\lambda,\sigma}f$ can be represented by the integral transform with the H-function in the kernel. Mapping properties such as the boundedness, the representation and the range of this transform on $\mathcal{L}_{\nu,r}$-spaces are established and its inversion formulas are given.

Key words: Mittag-Leffler integral transform, **H**-transform, Mittag-Leffler function, spaces of p-summable functions

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1. Introduction

The paper deals with the integral transform of the form

$$(\mathbf{E}_{\lambda,\sigma}f)(x) = \int_0^\infty E_{\lambda,\sigma}(-xt)f(t)dt \quad (x > 0) \tag{1.1}$$

with real positive $\lambda > 0$ and complex $\sigma \in \mathbf{C}$. The kernel of this transform contains the Mittag-Leffler function $E_{\lambda,\sigma}(z)$ defined for $\lambda > 0$ and $\sigma \in \mathbf{C}$ by [7, Section 18.1]

$$\mathbf{E}_{\lambda,\sigma}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\lambda k + \sigma)} \quad (z \in \mathbf{C}), \tag{1.2}$$

where $\Gamma(z)$ is the Euler gamma-function [6, Section 1]. The transform (1.1) is clearly defined for continuous functions $f \in C_0$ with compact support on $\mathbf{R}_+ = (0, \infty)$.

It should be noted that the interest to studying integral transforms with special function kernels was increased recently in connection with their applications in different problems of pure and applied mathematics, in this connection see the books by Brychkov and Prudnikov [2], Debnath [3] and Zayed [23]. Basically, they have considered integral transforms contain special bessel or hypergeometric functions in the kernels. The transforms (1.1) with the Mittag-Leffler function as the kernel were studied less. We can indicate only the book by Djrbashian [4] who considered the integral transforms in the form

$$(\mathbf{E}_{\lambda,\sigma;\theta}f)(x) = \int_0^\infty E_{\lambda,\sigma}\left((xt)^\lambda e^{\pm i\theta\lambda/2}\right) t^{\sigma-1} f(t)dt \quad (x > 0) \tag{1.3}$$

with positive $\lambda > 0$ and $\beta > 0$ and real θ. Djrbashian [4, Lemma 4.5] proved that if $0 < \lambda \leq 2$, $3/2 < \sigma < \alpha + 3/2$ and $\theta \in [(\lambda\pi)/2, 2\pi - (\lambda\pi)/2]$, then the transform $\mathbf{E}_{\lambda,\sigma;\theta}$ is bounded from $L_2(\mathbf{R}_+)$ into $\mathcal{L}_{2\sigma-1,2}$ (see (1.4)), established its inversion formula and some abelian theorems; the results of Djrbashian were also presented in his book [5, Section 1.6] and in the book by Zayed [23, Sections 16.2-16.4].

We study the Mittag-Leffler transform (1.1) in the spaces $\mathcal{L}_{\nu,r}$ of those complex-valued Lebesgue measurable functions f on $\mathbf{R}_+$ such that $||f||_{\nu,r} < \infty$, where

$$||f||_{\nu,r} = \left(\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t}\right)^{1/r} \quad (1 \leq r < \infty,\ \nu \in \mathbf{R} = (-\infty, \infty)) \tag{1.4}$$

and

$$||f||_{\nu,\infty} = ess\ sup_{x>0} x^\nu |f(x)|\ (\nu \in \mathbf{R}). \tag{1.5}$$

We note that, when $\nu = 1/r$ $(1 \leq r \leq \infty)$, the space $\mathcal{L}_{1/r,r}$ coincides with the classical $L^r(\mathbf{R}_+)$-space: $L^r(\mathbf{R}_+) = \mathcal{L}_{1/r,r}$.

We investigate the mapping properties such as the boundedness, the representation, the range and the inversion of the Mittag-Leffler transform $\mathbf{E}_{\lambda,\sigma}$ on $\mathcal{L}_{\nu,r}$-spaces.

We show that the Mittag-Leffler transform (1.1) is a special case of the so-called $\mathbf{H}$- transfrom

$$(\mathbf{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q\beta_q)\end{matrix}\right] f(t)dt \tag{1.6}$$

with the H-function as kernel - see, for example, [15, Chapter 2], [16, Section 8.3] and [22, Chapter 1]. This transform has the property

$$(\mathcal{M}\mathbf{H}f)(s) = \mathcal{H}(s)(\mathcal{M}f)(1-s), \tag{1.7}$$

with

$$\mathcal{H}(s) = \mathcal{H}_{p,q}^{m,n}\left[\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1,\cdots,(b_q\beta_q)\end{matrix} \,\middle|\, s\right] =$$

$$= \left(\mathcal{M}H_{p,q}^{m,n}\left[t \,\middle|\, \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q\beta_q)\end{matrix}\right]\right)(s) =$$

$$= \frac{\prod_{j=1}^m \Gamma(b_j+\beta_j s)\prod_{i=1}^n \Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i+\alpha_i s)\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j s)} \tag{1.8}$$

under certain conditions on the function f. Here m, n, p, q are integers such that $0 \leq m \leq q$, $0 \leq n \leq p$; a_i, $b_j \in \mathbf{C}$; α_i, $\beta_j \in \mathbf{R}$ $(1 \leq i \leq p,\ 1 \leq j \leq q)$ and empty products, if they occur, are taken to be one. $\mathcal{M}$ is the Mellin transform defined by

$$(\mathcal{M}f)(s) = \int_0^\infty f(t)t^{s-1}dt. \tag{1.9}$$

We note that for $f \in \mathcal{L}_{\nu,r}$ with $1 \leq r \leq 2$ the Mellin transform $\mathcal{M}$ is defined by

$$(\mathcal{M}f)(s) = \int_{-\infty}^\infty e^{(\sigma+it)\tau} f(e^\tau)d\tau \ (s = \sigma + it,\ \sigma, t \in \mathbf{R}), \tag{1.10}$$

and if $f \in \mathcal{L}_{\nu,r} \bigcap \mathcal{L}_{\nu,1}$ and $Re(s) = \nu$, then (1.10) coincides with (1.9), see [17].

Mapping properties such as the boundedness, the representation and the range of the **H**-transform (1.6) were proved independently in [8]-[9], [11]-[13] and [1], while the invertibility of (1.6) in $\mathcal{L}_{\nu,r}$ was given in [19]; in this connection see also [10, Sections 3 and 4].

In this paper we apply the results in [8] and [19] to investigate such properties of $\mathbf{E}_{\lambda,\sigma}$-transform (1.1). Section 2 deals with some results from the $\mathcal{L}_{\nu,r}$-theory of the **H**-transfrom (1.6). In Section 3 we discuss the Mittag-Leffler transform as the **H**-transform. Section 4 is devoted to the boundedness, the range and the representation of the Mittag-Leffler transform $\mathbf{E}_{\lambda,\sigma}$ in the space $\mathcal{L}_{\nu,r}$ for $r = 2$ and any $r \geq 1$. Section 5 deals with the inversion of the transform $\mathbf{E}_{2,\sigma}$ in $\mathcal{L}_{\nu,r}$ $(r \geq 1)$.

It should be noted that we can investigate the mapping properties of the Mittag-Leffler transform $\mathbf{E}_{\lambda,\sigma}$ in $\mathcal{L}_{\nu,r}$ with $0 < \lambda \leq 2$, and that the results will be different in the cases $0 < \lambda < 1$, $\lambda = 1$, $1 < \lambda < 2$ and $\lambda = 2$. Moreover, we can construct the inversion of such a transform in the frame of this space only for the operator $\mathbf{E}_{\lambda,\sigma}$ with $\lambda = 2$:

$$(\mathbf{E}_{2,\sigma}f)(x) = \int_0^\infty E_{2,\sigma}(-xt)f(t)dt \quad (x > 0;\ \sigma \in \mathbf{C}). \tag{1.11}$$

Therefore the problem to construct $\mathcal{L}_{\nu,r}$-theory of the Mittag-Leffler transform (1.1) for $\lambda > 2$ stay open as well as the problem of its inversion for $\lambda \neq 2$.

We also indicate that recently interest to the Mittag-Leffler function (1.2) and some of their modifications and generalizations is increased by their applications in various physical and mathematical problems, in particular in the framework of fractional differential equations; for example, see the survey papers [10] and [14]. In this connection we hope that the results obtained in this paper can be applied to the study of some problems in pure and applied mathematics by analogy with the investigations carried out by using Hankel-type integral transforms; for example, see the books by Sneddon [20]-[21].

2. Auxiliary Results

In this section we give some results from the theory of the the **H**-transform (1.6) on $\mathcal{L}_{\nu,r}$-spaces given in [9] and [19]. Following these papers we use the notation

$$\alpha = \begin{cases} \max\left[-\frac{Re(b_1)}{\beta_1}, \cdots, -\frac{Re(b_m)}{\beta_m}\right] \ (m > 0), \\ -\infty \ (m = 0); \end{cases} \tag{2.1}$$

$$\beta = \begin{cases} \min\left[\frac{1-Re(a_1)}{\alpha_1}, \cdots, \frac{1-Re(a_n)}{\alpha_n}\right] \ (n > 0), \\ \infty \ (n = 0); \end{cases} \tag{2.2}$$

$$a^\star = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j; \tag{2.3}$$

$$a_1^\star = \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i, \ a_2^\star = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j; \tag{2.4}$$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \tag{2.5}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}. \tag{2.6}$$

We denote by $E_{\mathcal{H}}$ the exceptional set of the function $\mathcal{H}$ defined in (1.8) which is the set of real numbers ν such that $\alpha < 1-\nu < \beta$ and $\mathcal{H}(s)$ has a zero on the line $Re(s) = 1-\nu$. We denote by $[X, Y]$ the collection of bounded linear operators from a Banach space X into a Banach space Y.

The $\mathcal{L}_{\nu,2}$-theory of the $\mathbf{H}$-transform is given by two following statements.

Theorem A ([9, Theorem 3].) *Let $\alpha < 1-\nu < \beta$ and either $a^\star > 0$ or $a^\star = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) \leq 0$. Then the following assertions hold:*

(a) *There is one-to-one transform $\mathbf{H} \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$ so that the relation (1.7) holds for $f \in \mathcal{L}_{\nu,2}$ and $\mathrm{Re}(s) = 1-\nu$. If $a^\star = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) = 0$ and $\nu \notin E_{\mathcal{H}}$, then the operator $\mathbf{H}$ maps $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$.*

(b) *For $f, g \in \mathcal{L}_{\nu,2}$ the relation of integration by parts holds*

$$\int_0^\infty f(x)(\mathbf{H}g)(x)dx = \int_0^\infty g(x)(\mathbf{H}f)(x)dx. \tag{2.7}$$

(c) *Let $f \in \mathcal{L}_{\nu,2}$, $w \in \mathbf{C}$ and $h > 0$. If $\mathrm{Re}(w) > (1-\nu)h - 1$, then $\mathbf{H}f$ is given by*

$$(\mathbf{H}f)(x) = x^{1-(w+1)/h} \frac{d}{dx} x^{(w+1)/h}$$

$$\times \int_0^\infty H^{m,n+1}_{p+1,q+1} \left[xt \left| \begin{matrix} (-w,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ (b_1,\beta_1),\cdots,(b_q\beta_q),(-w-1,h) \end{matrix} \right. \right] f(t)dt. \tag{2.8}$$

If $\mathrm{Re}(w) < (1-\nu)h - 1$, then

$$(\mathbf{H}f)(x) = -x^{1-(w+1)/h} \frac{d}{dx} x^{(w+1)/h}$$

$$\times \int_0^\infty H^{m+1,n}_{p+1,q+1} \left[xt \left| \begin{matrix} (a_1,\alpha_1),\cdots,(a_p,\alpha_p),(-w,h) \\ (-w-1,h)(b_1,\beta_1),\cdots,(b_q\beta_q) \end{matrix} \right. \right] f(t)dt. \tag{2.9}$$

(d) *$\mathbf{H}$ is independent on ν in the sense if ν_1 and ν_2 satisfy the conditions in the first sentence of this theorem, and if the transforms $\mathbf{H}_1$ and $\mathbf{H}_2$ are defined in $\mathcal{L}_{\nu_1,2}$ and $\mathcal{L}_{\nu_2,2}$ respectively by (1.7), then $\mathbf{H}_1 f = \mathbf{H}_2 f$ for $f \in \mathcal{L}_{\nu_1,2} \bigcap \mathcal{L}_{\nu_2,2}$.*

Theorem B ([9, Theorem 4].) *Let $\alpha < 1-\nu < \beta$, $f \in \mathcal{L}_{\nu,2}$ and either $a^\star > 0$ or $a^\star = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) < 0$. Then for $x \in \mathbf{R}_+$ the transform $(\mathbf{H}f)(x)$ is given by (1.6).*

Some of the results above can be extended to the space $\mathcal{L}_{\nu,r}$ with any $r \in [1,\infty)$. The range of the $\mathbf{H}$-transform on $\mathcal{L}_{\nu,r}$ is different in nine cases. We shall use only four cases; a) $a^\star = 0$ and $\Delta > 0$; b) $a_1^\star > 0$ and $a_2^\star > 0$; c) $a_1^\star > 0$ and $a_2^\star = 0$ and d) $a^\star > 0$, $a_1^\star > 0$ and $a_2^\star < 0$. For this we need the elementary transform M_ξ of the forms

$$(M_\xi f)(x) = x^\xi f(x) \ (\xi \in \mathbf{C}), \tag{2.10}$$

the Erdelyi-Kober fractional integration operator $I^\alpha_{-;\delta,\eta}$ (see [18, Section 18.1]) defined for $\alpha, \eta \in \mathbf{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\delta \in \mathbf{R}_+$ by

$$(I^\alpha_{-;\delta,\eta} f)(x) = \frac{\delta x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^\infty (t^\delta - x^\delta)^{\alpha-1} t^{\delta(1-\alpha-\eta)-1} f(t)dt \quad (x > 0), \tag{2.11}$$

the modified Hankel transform $H_{k,\eta}$ given for $k \in \mathbf{R}$ ($k \neq 0$) and $\eta \in \mathbf{C}$ with $\mathrm{Re}(\eta) > -1$ by

$$(H_{k,\eta} f)(x) = \int_0^\infty (xt)^{1/k-1/2} J_\eta\left(|k|)(xt)^{1/k}\right) f(t)dt \ (x > 0); \tag{2.12}$$

and the modified Laplace transform $L_{k,\xi}$ defined for $k \in \mathbf{R}$ ($k \neq 0$) and $\xi \in \mathbf{C}$ by

$$(L_{k,\xi} f)(x) = \int_0^\infty (xt)^{-\xi} \exp\left(-|k|(xt)^{1/k}\right) f(t)dt \quad (x > 0). \tag{2.13}$$

Note that if $k = 1$, the transform (2.12) coincides with the classical Hankel transform

$$(H_{1,\eta} f)(x) \equiv (H_\eta f)(x) = \int_0^\infty (xt)^{1/2} J_\eta(xt) f(t)dt \ (x > 0), \tag{2.14}$$

while for $k = 1$ and $\xi = 0$ (2.13) yields the Laplace transform

$$(L_{1,0} f)(x) \equiv (Lf)(x) = \int_0^\infty e^{-xt} f(t)dt \quad (x > 0). \tag{2.15}$$

There hold the following assertions.

Theorem C ([9, Theorem 7]). *Let* $a^\star = 0$, $\Delta > 0$, $\alpha < 1 - \nu < \beta$, $1 < r < \infty$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leq 1/2 - \gamma(r)$, *where*

$$\gamma(r) = \max\left[\frac{1}{r}, \frac{1}{r'}\right], \quad \frac{1}{r} + \frac{1}{r'} = 1. \tag{2.16}$$

(a) *The* $\mathbf{H}$*-transform defined on* $\mathcal{L}_{\nu,2}$ *can be extended to* $\mathcal{L}_{\nu,r}$ *as an element of* $[\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,\rho}]$ *for all* ρ *with* $r \leq \rho < \infty$ *such that* $\rho' \geq [1/2 - \Delta(1-\nu) - \mathrm{Re}(\mu)]^{-1}$.

(b) If $1 \leq r \leq 2$, *then* $\mathbf{H}$ *is a one-to-one transform on* $\mathcal{L}_{\nu,r}$ *and there holds the equality (1.7).*

(c) *If* $f \in \mathcal{L}_{\nu,r}$ *and* $g \in \mathcal{L}_{\nu,\rho}$, $1 < \rho < \infty$, $1/r + 1/\rho \geq 1$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leq 1/2 - \max[\gamma(r), \gamma(\rho)]$, *then the relation (2.7) holds.*

(d) *If* $\nu \notin \mathcal{E}_\mathcal{H}$, *then the transform* $\mathbf{H}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$. *If we set* $\xi = -\Delta\alpha - \mu - 1$, *then* $\mathrm{Re}(\xi) > -1$ *and there holds*

$$\mathbf{H}(\mathcal{L}_{\nu,r}) = \left(M_{\mu/\Delta+1/2} H_{\Delta,\xi}\right)\left(\mathcal{L}_{\nu-1/2-\mathrm{Re}(\mu)/\Delta,r}\right). \tag{2.17}$$

When $\nu \in \mathcal{E}_\mathcal{H}$, *then* $\mathbf{H}(\mathcal{L}_{\nu,r})$ *is a subset of the right hand side of (2.17).*

(e) *If* $f \in \mathcal{L}_{\nu,r}$, $w \in \mathbf{C}$, $h > 0$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leq 1/2 - \gamma(r)$, *then* $\mathbf{H}f$ *is given in (2.8) for* $\mathrm{Re}(w) > (1-\nu)h - 1$, *and by (2.9) for* $\mathrm{Re}(w) < (1-\nu)h - 1$. *If* $\Delta(1-\nu) + \mathrm{Re}(\nu) < 0$, $\mathbf{H}f$ *is given by (1.6).*

Theorem D ([9, Theorem 9]). *Let* $a^\star > 0$, $\alpha < 1 - \nu < \beta$ *and* $1 \leq r \leq \rho \leq \infty$.

(a) *The* $\mathbf{H}$*-transform defined on* $\mathcal{L}_{\nu,2}$ *can be extended to* $\mathcal{L}_{\nu,r}$ *as an element of* $[\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,\rho}]$. *If* $1 \leq r \leq 2$, *then* $\mathbf{H}$ *is a one-to-one transform from* $\mathcal{L}_{\nu,r}$ *onto* $\mathcal{L}_{1-\nu,s}$.

(b) *If* $f \in \mathcal{L}_{\nu,r}$ *and* $g \in \mathcal{L}_{\nu,\rho'}$ *with* $(1/\rho + 1/\rho' = 1$, *then the relation (2.7) holds.*

Theorem E ([9, Theorem 10]). *Let* $a_1^\star > 0$, $a_2^\star > 0$, $m > 0$, $n > 0$, $\alpha < 1-\nu < \beta$ *and* $\omega = \mu + a_1^\star\alpha - a_2^\star\beta + 1$ *and let* $1 < r < \infty$.

(a) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leq 2$, *then the transform* $\mathbf{H}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$.

(b) *If* $\mathrm{Re}(\omega) \geq 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\mathbf{H}\left(\mathcal{L}_{\nu,r}\right) = \left(L_{a_1^\star,\alpha} L_{a_2^\star,1-\beta-\omega/a_2^\star}\right)\left(\mathcal{L}_{1-\nu,r}\right). \tag{2.18}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\mathbf{H}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (2.18).*

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\mathbf{H}\left(\mathcal{L}_{\nu,r}\right) = \left(I^{-\omega}_{-;1/a_1^\star,-a_1^\star\alpha} L_{a_1^\star,\alpha} L_{a_2^\star,1-\beta}\right)\left(\mathcal{L}_{1-\nu,r}\right). \tag{2.19}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\mathbf{H}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (2.19).*

Theorem F ([9, Theorem 11]). *Let* $a_1^\star > 0$, $a_2^\star = 0$, $m > 0$, $\alpha < 1-\nu < \beta$ *and* $\omega = \mu + a_1^\star\alpha + 1/2$ *and let* $1 < r < \infty$.

(a) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leq 2$, *then the transform* $\mathbf{H}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$.

(b) *If* $\mathrm{Re}(\omega) \geq 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$. *then*

$$\mathbf{H}\left(\mathcal{L}_{\nu,r}\right) = \left(L_{a_1^\star,\alpha-\omega/a_1^\star}\right)\left(\mathcal{L}_{\nu,r}\right). \tag{2.20}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\mathbf{H}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (2.20).*

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\mathbf{H}\left(\mathcal{L}_{\nu,r}\right) = \left(I^{-\omega}_{-;1/a_1^\star,-a_1^\star\alpha} L_{a_1^\star,\alpha}\right)\left(\mathcal{L}_{\nu,r}\right). \tag{2.21}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\mathbf{H}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (2.21).*

Theorem G ([9, Theorem 13]). *Let* $a^* > 0$, $a_1^\star > 0$, $a_2^\star < 0$, $\alpha < 1-\nu < \beta$ *and* $1 < r < \infty$.

(a) *If either* $\nu \notin E_{\mathcal{H}}$ *or* $1 < r \leq 2$, *then* $\mathbf{H}$ *is a one-to-one on* $\mathcal{L}_{\nu,r}$.

(b) *Let*

$$\omega = a^*\xi - \mu - \frac{1}{2}, \tag{2.22}$$

where μ *is given by (2.6), and let* $\xi, \zeta \in \mathbf{C}$ *be chosen as*

$$a^*\mathrm{Re}(\xi) \geq \gamma(r) + 2a_2^*(\nu - 1) + \mathrm{Re}(\mu),\ \ \mathrm{Re}(\xi) > \nu - 1 \tag{2.23}$$

$$\mathrm{Re}(\zeta) < 1 - \nu. \tag{2.24}$$

If $\nu \notin E_{\mathcal{H}}$, *then*

$$\mathbf{H}(\mathcal{L}_{\nu,r}) = \left(M_{1/2+\omega/(2a_2^\star)} H_{-2a_2^\star,2a_2^\star\zeta+\omega-1} L_{-a^\star,1/2+\xi-\omega/(2a_2^\star)}\right)\left(\mathcal{L}_{3/2-\nu+\mathrm{Re}(\omega)/(2a_2^\star),r}\right). \tag{2.25}$$

When $\nu \in E_{\mathcal{H}}$, *then* $\mathbf{H}(\mathcal{L}_{\nu,r})$ *is a subset of the right hand side of (2.25).*

3. Mittag-Leffler transform as the H-transform

The $\mathcal{L}_{\nu,r}$-theory of the Mittag-Leffler transform (1.1) is based on the Mellin transform of the Mittag-Leffler function (1.2).

Lemma 1. *Let* $\lambda, \sigma \in \mathbf{R}$ *and* $s \in \mathbf{C}$ *be such that* $0 < \lambda \leq 2$, $3/2 < \sigma < \lambda + 3/2$ *and* $\lambda\mathrm{Re}(s) = \sigma - 3/2$.

Then the Mellin transform (1.9) of $E_{\lambda,\sigma}(-x)$ is given by

$$(\mathcal{M}[E_{\lambda,\sigma}(-x)])(s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\sigma-\lambda s)}. \tag{3.1}$$

Proof. It is known the relation [4, Lemma 5]:

$$\int_0^\infty E_{1/\varrho,\mu+1}\left(xe^{i\alpha}\right)x^{\varrho(s+\mu-1)-1}dx = \frac{\pi}{\Gamma(2-s)}\frac{e^{i\varrho(\pi-\alpha)(s+\mu-1)}}{\sin[\pi\varrho(s+\mu-1)]} \quad \left(\mathrm{Re}(s)=\frac{1}{2}\right), \tag{3.2}$$

being held for

$$\varrho \geq \frac{1}{2},\ \frac{1}{2} < \mu < \frac{1}{2}+\frac{1}{\varrho},\ \frac{\pi}{2\varrho} \leq \alpha \leq 2\pi - \frac{\pi}{2\varrho}. \tag{3.3}$$

Taking $\alpha=\pi$, $\varrho=1/\lambda$, $\mu=\sigma-1$ and replacing $\varrho(s+\mu-1)$ by s, the conditions in (3.2) transfer to the condition of Lemma, and (3.3) is reduced to the relation

$$\int_0^\infty E_{\lambda,\sigma}(-x)x^{s-1} = \frac{\pi}{\Gamma(\sigma-\lambda s)}\frac{1}{\sin(\pi s)} \quad \left(\mathrm{Re}(s\lambda-\sigma+2)=\frac{1}{2}\right). \tag{3.4}$$

In accordance with the reflection formula for the gamma-function (see [6, 1.2(6)])

$$\frac{1}{\sin(\pi s)} = \frac{\Gamma(s)\Gamma(1-s)}{\pi}$$

and hence (3.4) yields (3.1). Thus lemma is proved.

The relation (3.1) can be extended to more general complex σ and s.

Lemma 2. *Let $\lambda \in \mathbf{R}$, $\sigma \in \mathbf{R}$ and $s \in \mathbf{C}$ and let either of the conditions* **(a)** $0<\lambda<2$, $0<\mathrm{Re}(s)<1$ *or* **(b)** $\lambda=2$, $0<\mathrm{Re}(\beta)<3$, $0<\mathrm{Re}(s)<\min[1,\mathrm{Re}(\sigma)/2]$ *hold.*

Then the Mellin transform (1.9) of $E_{\lambda,\sigma}(-x)$ is given by (3.1).

Proof. We use analytic continuation to extend (3.1) to the range of complex β and s indicated in (a) and (b). Since $\Gamma(z)$ is analytic function of $z \in \mathbf{C}$ having simple poles at the points $z=0,-1,-2,\cdots$ [6, Section 1], the right hand side of (3.1) is analytic function of $s \in \mathbf{C}$ except points $s=0,\pm1,\pm2,\cdots$. The left hand side of (3.1) is also analytic of those $s \in \mathbf{C}$ for which the integral

$$(\mathcal{M}[E_{\lambda,\sigma}(-x)])(s) = \int_0^\infty E_{\lambda,\sigma}(-x)]x^{s-1}dx \tag{3.5}$$

converges. By (1.2)

$$E_{\lambda,\sigma}(-x)x^{s-1} = O\left(\frac{1}{x^{1-s}}\right) \quad (x\to+0)$$

and the integral (3.5) converges at zero for $\mathrm{Re}(s)>0$. According to [4, Lemmas 3.4 and 3.5], the Mittag-Leffler function (1.2) has different asymptotic behavior at infinity for $0<\lambda<2$ and $\lambda=2$, $0<\mathrm{Re}(\beta)<3$:

$$E_{\lambda,\sigma}(-x) = O\left(\frac{1}{x}\right) \quad (x\to+\infty;\ \ 0<\lambda<2)$$

and

$$E_{\lambda,\sigma}(-x) = x^{(1-\sigma)/2}\cos\left[\sqrt{x}+\frac{\pi}{2}(1-\sigma)\right] + O\left(\frac{1}{x}\right) \quad (x\to+\infty;\ \ \lambda=2),$$

respectively. Therefore

$$E_{\lambda,\sigma}(-x)x^{s-1} = O\left(\frac{1}{x^{2-s}}\right) \quad (x\to+\infty;\ \ 0<\lambda<2)$$

and

$$E_{\lambda,\sigma}(-x)x^{s-1} = \frac{1}{x^{-s+(1+\sigma)/2}} \cos\left[\sqrt{x} + \frac{\pi}{2}(1-\sigma)\right] + O\left(\frac{1}{x^{2-s}}\right) \quad (x \to +\infty; \quad \lambda = 2),$$

and, in accordance with the known convergence theorems of analysis, the integral in left hand side of (3.5) converges at infinity for $0 < \lambda < 2$, $\mathrm{Re}(s) < 1$ and for $\lambda = 2$, $0 < \mathrm{Re}(\beta) < 3$, $\mathrm{Re}(s) < \min[1, \mathrm{Re}(\sigma)/2]$, and these conditions are satisfied by the conditions of Lemma. Therefore the analytic continuation yields the result in lemma.

Remark 1. The relation (3.1) was indicated in [16] but without the condition $0 < \mathrm{Re}(\beta) < 3$ in the case $\lambda = 2$.

Using (3.1), it is directly verified for "sufficiently good" function f that the Mellin transform $\mathcal{M}$ of (1.1) is given by

$$(\mathcal{M}\mathbf{H}f)(s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\sigma-\lambda s)}(\mathcal{M}f)(1-s). \tag{3.6}$$

According to (1.8) the function in the right hand side of (3.1) is the $\mathcal{H}$-function

$$\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\sigma-\lambda s)} = \mathcal{H}^{1,1}_{1,2}\left[\begin{array}{c} (0,1) \\ (0,1),(1-\sigma,\lambda) \end{array} \middle| s\right]. \tag{3.7}$$

Therefore (3.6) is the relation of the form (1.7), and hence the $\mathbf{E}_{\lambda,\sigma}f$ -transform (1.1) is a special $\mathbf{H}$-transform (1.6):

$$(\mathbf{E}_{\lambda,\sigma}f)(x) = \int_0^\infty H^{1,1}_{1,2}\left[xt \middle| \begin{array}{c} (0,1) \\ (0,1),(1-\sigma,\lambda) \end{array}\right] f(t)dt. \tag{3.8}$$

According to (2.1)-(2.6) we have

$$\alpha = 0, \quad \beta = 1, \quad a^* = 2-\lambda, \quad a_1^* = 1, \quad a_2^* = 1-\lambda, \quad \Delta = \lambda, \quad \mu = \frac{1}{2} - \sigma. \tag{3.9}$$

If $\mathcal{E}_{\mathcal{H}}$ is the exceptional set of the $\mathcal{H}$-function in (3.7), then ν is not in the exceptional set $\mathcal{E}_{\mathcal{H}}$, if

$$s \neq \frac{\sigma+k}{\lambda} \quad (k = 0,1,2,\cdots). \tag{3.10}$$

4. $\mathcal{L}_{\nu,r}$-theory of the Mittag-Leffler transform

Using the results in the previous section, we can apply Theorems A-G to construct the $\mathcal{L}_{\nu,r}$-theory of the Mittag-Leffler transform (1.1). First from Theorems A and B we deduce the $\mathcal{L}_{\nu,2}$-theory of this transform.

Theorem 1. *Let $0 < \nu < 1$ and let $\alpha > 0$ and $\sigma \in \mathbf{C}$ be such that either $0 < \lambda < 2$ or $\lambda = 2$, $0 < \mathbf{Re}(\sigma) < 3$, $2\nu + \mathrm{Re}(\sigma) \geq 5/2$. Then the following assertions hold:*

(a) *There is one-to-one transform $\mathbf{E}_{\lambda,\sigma} \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$ so that the relation (3.6) holds for $f \in \mathcal{L}_{\nu,2}$ and $\mathrm{Re}(s) = 1-\nu$. If $\lambda = 2$, $0 < \mathbf{Re}(\sigma) < 3$, $2\nu + \mathrm{Re}(\sigma) = 5/2$ and the condition in (3.10) is fulfilled, then the operator $\mathbf{E}_{\lambda,\sigma}$ maps $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$.*

(b) *For $f, g \in \mathcal{L}_{\nu,2}$ the relation of integration by parts holds*

$$\int_0^\infty f(x)(\mathbf{E}_{\lambda,\sigma}g)(x)dx = \int_0^\infty g(x)(\mathbf{E}_{\lambda,\sigma}f)(x)dx. \tag{4.1}$$

(c) *Let $f \in \mathcal{L}_{\nu,2}$, $w \in \mathbf{C}$ and $h > 0$. If $\mathrm{Re}(w) > (1-\nu)h - 1$, then $\mathbf{E}_{\lambda,\sigma}f$ is given by*

$$(\mathbf{E}_{\lambda,\sigma}f)(x) = x^{1-(w+1)/h}\frac{d}{dx}x^{(w+1)/h}$$

$$\times \int_0^\infty H_{2,3}^{1,2}\left[xt \left| \begin{matrix} (-w,h),(0,1) \\ (0,1),(1-\sigma,\lambda),(-w-1,h) \end{matrix} \right.\right] f(t)dt. \tag{4.2}$$

If $\mathrm{Re}(w) < (1-\nu)h - 1$, *then*

$$(\mathbf{E}_{\lambda,\sigma}f)(x) = -x^{1-(w+1)/h}\frac{d}{dx}x^{(w+1)/h}$$

$$\times \int_0^\infty H_{2,3}^{2,1}\left[xt \left| \begin{matrix} (0,1),(-w,h) \\ (-w-1,h),(0,1),(1-\sigma,\lambda) \end{matrix} \right.\right] f(t)dt. \tag{4.3}$$

(d) $\mathbf{E}_{\lambda,\sigma}$ *is independent on* ν *in the sense if* ν_1 *and* ν_2 *satisfy the conditions in the first sentence of this theorem, and if the transforms* $\mathbf{E}_{\lambda,\sigma;1}$ *and* $\mathbf{E}_{\lambda,\sigma;2}$ *are defined in* $\mathcal{L}_{\nu_1,2}$ *and* $\mathcal{L}_{\nu_2,2}$ *respectively by (3.6), then* $\mathbf{E}_{\lambda,\sigma;1}f = \mathbf{E}_{\lambda,\sigma;2}f$ *for* $f \in \mathcal{L}_{\nu_1,2}\bigcap\mathcal{L}_{\nu_2,2}$.

(e) *If* $f \in \mathcal{L}_{\nu,2}$ *and either* $0 < \lambda < 2$ *or* $\lambda = 2$, $0 < \mathbf{Re}(\sigma) < 3$, $2\nu + \mathrm{Re}(\sigma) > 5/2$, *then for* $x \in \mathbf{R}_+$ *the transform* $(\mathbf{E}_{\lambda,\sigma}f)(x)$ *is given by (1.1) and (3.8).*

If $\lambda = 2$, then according to (3.9) $a^* = 0$ and from Theorem C we deduce $\mathcal{L}_{\nu,r}$-theory of the Mittag-Leffler transform (1.1) for any $r > 1$.

Theorem 2. *Let* $\lambda = 2$, $0 < \mathrm{Re}(\sigma) < 3$, $0 < \nu < 1$ *and* $1 < r < \infty$ *be such that* $2\nu + \mathrm{Re}(\sigma) \geq 2 + \gamma(r)$, *where* $\gamma(r)$ *is given in (2.16).*

(a) *The* $\mathbf{E}_{2,\sigma}$*-transform defined on* $\mathcal{L}_{\nu,2}$ *can be extended to* $\mathcal{L}_{\nu,r}$ *as an element of* $[\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,\rho}]$ *for all* ρ *with* $r \leq \rho < \infty$ *such that* $\rho' \geq [2\nu - \mathrm{Re}(\sigma) - 2]^{-1}$.

(b) *If* $1 \leq r \leq 2$, *then* $\mathbf{E}_{2,\sigma}$ *is a one-to-one transform on* $\mathcal{L}_{\nu,r}$ *and there holds the equality (3.6) with* $\lambda = 2$.

(c) *If* $f \in \mathcal{L}_{\nu,r}$ *and* $g \in \mathcal{L}_{\nu,\rho}$, $1 < \rho < \infty$, $1/r + 1/\rho \geq 1$ *and* $2\nu + \mathrm{Re}(\sigma) \geq 2 + \max[\gamma(r), \gamma(\rho)]$, *then the relation (4.1) with* $\lambda = 2$ *holds.*

(d) *If the condition in (3.10) with* $\lambda = 2$ *is satisfied, then the transform* $\mathbf{E}_{2,\sigma}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$ *and and there holds*

$$\mathbf{E}_{2,\sigma}(\mathcal{L}_{\nu,r}) = \left(M_{3/4-\sigma/2}H_{2,\sigma-1/2}\right)\left(\mathcal{L}_{\nu-\mathrm{Re}(\sigma)-1,r}\right). \tag{4.4}$$

If the condition in (3.10) with $\lambda = 2$ *is not satisfied, then* $\mathbf{E}_{2,\sigma}(\mathcal{L}_{\nu,r})$ *is a subset of the right hand side of (4.4).*

(e) *If* $f \in \mathcal{L}_{\nu,r}$, $w \in \mathbf{C}$, $h > 0$ *and* $2\nu) + \mathrm{Re}(\sigma) \leq 2 + \gamma(r)$, *then* $\mathbf{E}_{2,\sigma}f$ *is given in (4.2) for* $\mathrm{Re}(w) > (1-\nu)h - 1$, *and by (4.3) for* $\mathrm{Re}(w) < (1-\nu)h - 1$ *with* $\lambda = 2$. *If* $2\nu + \mathrm{Re}(\sigma) > 5/2$, $\mathbf{E}_{2,\sigma}f$ *is given by (1.1) and (3.8) with* $\lambda = 2$.

Let now $0 < \lambda < 2$, and hence $a^* > 0$ by (3.9). From Theorem E we obtain the boundedness of the Mittag-Leffler transform (1.1) in $\mathcal{L}_{\nu,r}$ and the relation of the integration by parts.

Theorem 3. *Let* $0 < \lambda < 2$, $0 < \nu < 1$ *and* $1 \leq r \leq \rho \leq \infty$.

(a) *The* $\mathbf{E}_{\lambda,\sigma}$*-transform defined on* $\mathcal{L}_{\nu,2}$ *can be extended to* $\mathcal{L}_{\nu,r}$ *as an element of* $[\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,\rho}]$. *If* $1 \leq r \leq 2$, *then* $\mathbf{E}_{\lambda,\sigma}$ *is a one-to-one transform from* $\mathcal{L}_{\nu,r}$ *onto* $\mathcal{L}_{1-\nu,s}$.

(b) *If* $f \in \mathcal{L}_{\nu,r}$ *and* $g \in \mathcal{L}_{\nu,\rho'}$ *with* $(1/\rho + 1/\rho' = 1$, *then the relation (4.1) holds.*

When $0 < \lambda < 2$, then by (3.9) $a_1^* = 1$ and $a_2^* = 1 - \lambda$. Theorems E-G give the following results in which the range of the Mittag-Leffler transform $\mathbf{E}_{\lambda,\sigma}$ will be different in the cases $0 < \lambda < 1$, $\lambda = 1$ and $1 < \lambda < 2$ corresponding to the ones in $a_2^* > 0$, $a_2^* = 0$ and $a_2^* < 0$, respectively.

Theorem 4. *Let* $0 < \lambda < 1$, $0 < \nu < 1$, $\omega = \sigma - \lambda - 1/2$ *and let* $1 < r < \infty$.

(a) *If the condition in (3.10) is satisfied, or if* $1 < r \leq 2$, *then the transform* $\mathbf{E}_{\lambda,\sigma}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$.

(b) *If* $\operatorname{Re}(\sigma) \leq \lambda + 1/2$ *and the condition in (3.10) is satisfied, then*

$$\mathbf{E}_{\lambda,\sigma}\left(\mathcal{L}_{\nu,r}\right) = \left(LL_{1-\lambda,\omega/(1-\lambda)}\right)\left(\mathcal{L}_{1-\nu,r}\right). \tag{4.5}$$

When the condition in (3.10) is not satisfied, $\mathbf{E}_{\lambda,\sigma}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (4.5).*

(c) *If* $\operatorname{Re}(\sigma) > \lambda + 1/2$ *and the condition in (3.10) is fulfilled, then*

$$\mathbf{E}_{\lambda,\sigma}\left(\mathcal{L}_{\nu,r}\right) = \left(I^{\omega}_{-;1,0}LL_{1-\lambda,0}\right)\left(\mathcal{L}_{1-\nu,r}\right). \tag{4.6}$$

When the condition in (3.10) is not satisfied, $\mathbf{E}_{\lambda,\sigma}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (4.6).*

Theorem 5. *Let* $0 < \nu < 1$ *and* $1 < r < \infty$.

(a) *If the condition in (3.10) with* $\lambda = 1$ *is satisfied, or if* $1 < r \leq 2$, *then the transform* $\mathbf{E}_{1,\sigma}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$.

(b) *If* $\sigma \leq 1$ *and the condition in (3.10) with* $\lambda = 1$ *are satisfied, then*

$$\mathbf{E}_{1,\sigma}\left(\mathcal{L}_{\nu,r}\right) = \left(L_{1,\sigma-1}\right)\left(\mathcal{L}_{\nu,r}\right). \tag{4.7}$$

When the condition in (3.10) with $\lambda = 1$ *is not satisfied,* $\mathbf{E}_{1,\sigma}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (4.7).*

(c) *If* $\sigma > 1$ *and the condition in (3.10) with* $\lambda = 1$ *is fulfilled, then*

$$\mathbf{E}_{1,\sigma}\left(\mathcal{L}_{\nu,r}\right) = \left(I^{\sigma-1}_{-;1,0}L\right)\left(\mathcal{L}_{\nu,r}\right). \tag{4.8}$$

When the condition in (3.10) with $= \lambda = 1$ *is not satisfied,* $\mathbf{E}_{1,\sigma}\left(\mathcal{L}_{\nu,r}\right)$ *is a subset of the right hand side of (4.8).*

Theorem 6. *Let* $1 < \lambda < 2$, $a_2^* = 1 - \lambda$, $0 < \nu < 1$ *and* $1 < r < \infty$.

(a) *If the condition in (3.10) is satisfied, or if* $1 < r \leq 2$, *then the transform* $\mathbf{E}_{\lambda,\sigma}$ *is one-to-one on* $\mathcal{L}_{\nu,r}$.

(b) *Let*

$$\omega = (2-\lambda)\xi - \mu + \sigma - 1, \tag{4.9}$$

and let $\xi, \zeta \in \mathbf{C}$ *be chosen as*

$$(2-\lambda)\operatorname{Re}(\xi) \geq \gamma(r) + 2(1-\lambda)(\nu-1) - \operatorname{Re}(\sigma) + \frac{1}{2}, \quad \operatorname{Re}(\xi) > \nu - 1 \tag{4.10}$$

$$\operatorname{Re}(\zeta) < 1 - \nu. \tag{4.11}$$

If the condition in (3.10) is satisfied, then

$$\mathbf{E}_{\lambda,\sigma}(\mathcal{L}_{\nu,r}) = \left(M_{1/2+\omega/(2a_2^*)}H_{-2a_2^*,2a_2^*\zeta+\omega-1}L_{\lambda-2,1/2+\xi-\omega/(2a_2^*)}\right)\left(\mathcal{L}_{3/2-\nu+\operatorname{Re}(\omega)/(2a_2^*),r}\right). \tag{4.12}$$

When the condition in (3.10) is not satisfied , then $\mathbf{E}_{\lambda,\sigma}(\mathcal{L}_{\nu,r})$ *is a subset of the right hand side of (4.12).*

5. Inversion of the Mittag-Leffler transform

On the basis of the results of Section 2, we deduce the inversion of the Mittag-Leffler transform (1.1) from the inversion of the **H**-transform (1.6). As it was indicated in Introduction, the inversion of the **H**-transform (1.6) in the spaces $\mathcal{L}_{\nu,r}$ was investigated in [19]. Here we present some results from this paper. Let

$$\alpha_0 = \begin{cases} \max\left[-\frac{Re(b_{m+1})-1}{\beta_{m+1}}+1,\cdots,-\frac{Re(b_q)-1}{\beta_q}+1\right] \ (q>m), \\ -\infty \ (q=m); \end{cases} \tag{5.1}$$

$$\beta_0 = \begin{cases} \min\left[\frac{Re(a_{n+1})}{\alpha_{n+1}}+1,\cdots,\frac{Re(a_p)}{\alpha_p}+1\right] \ (p>n), \\ \infty \ (p=n). \end{cases} \tag{5.2}$$

The inversion of the **H**-transform can have the respective form (2.8) or (2.9):

$$f(x) = x^{1-(w+1)/h}\frac{d}{dx}x^{(w+1)/h}$$

$$\times \int_0^\infty H_{p+1,q+1}^{q-m,p-n+1}\left[xt \left| \begin{matrix} (-w,h),(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n} \\ (1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m},(-w-1,h) \end{matrix} \right.\right](\mathbf{H}f)(t)dt \tag{5.3}$$

or

$$f(x) = -x^{1-(w+1)/h}\frac{d}{dx}x^{(w+1)/h}$$

$$\times \int_0^\infty H_{p+1,q+1}^{q-m+1,p-n}\left[xt \left| \begin{matrix} (1-a_i-\alpha_i,\alpha_1)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n},(-w,h) \\ (-w-1,h)(1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m} \end{matrix} \right.\right](\mathbf{H}f)(t)dt. \tag{5.4}$$

These formulas are true provided that $a^* = 0$ under some additional conditions which are different in the cases $\Delta = 0$, $\Delta > 0$ and $\Delta < 0$. We give only the result for $\Delta > 0$.

Theorem H ([19, Theorem 4.1]). *Let* $1 < r < \infty$, $-\infty < \alpha < 1-\nu < \beta$, $\alpha_0 < \nu < \min\{\beta_0, [\mathrm{Re}(\mu+1/2)/\Delta]+1\}$, $a^* = 0$, $\Delta > 0$ *and* $\Delta(1-\nu)+\mathrm{Re}(\mu) \leq 1/2+\gamma(r)$, *where* $\gamma(r)$ *is given by (2.16). Let* $w \in \mathbf{C}$ *and* $h > 0$.

If $f \in \mathcal{L}_{\nu,r}$, *then the relation (5.3) holds for* $\mathrm{Re}(w) > \nu h - 1$, *while the formula (5.4) is valid for* $\mathrm{Re}(w) < \nu h - 1$.

According to (3.8), (3.9), (5.1) and (5.2)

$$\alpha = 0, \quad \beta = 1, \quad a^* = 2-\lambda, \quad \mu = \frac{1}{2}-\sigma, \quad \alpha_0 = 1-\frac{\mathrm{Re}(\sigma)}{\lambda}, \quad \beta_0 = \infty. \tag{5.5}$$

Since $a^* = 0$ for $\lambda = 2$, then from Theorem H we deduce the following result which yields the inversion of the Mittag-Leffler transform (1.11) in the space $\mathcal{L}_{\nu,r}$.

Theorem 7. *Let* $1 < r < \infty$, $\nu \in \mathbf{R}$ *and* $\sigma \in \mathbf{C}$ *be such that*

$$\max\left[0, 1-\frac{\mathrm{Re}(\sigma)}{2}\right] < \nu < \min\left[1, \frac{3}{2}-\frac{\mathrm{Re}(\sigma)}{2}\right], \quad 2\nu+\mathrm{Re}(\sigma) \geq 2+\gamma(r). \tag{5.6}$$

If $f \in \mathcal{L}_{\nu,r}$ *and* $\mathrm{Re}(w) > \nu h - 1$, *then the inversion formula*

$$f(x) = x^{1-(w+1)/h}\frac{d}{dx}x^{(w+1)/h}$$

$$\times \int_0^\infty H_{2,3}^{1,1}\left[xt \left| \begin{matrix} (-w,h),(0,1) \\ (\sigma-2,2),(0,1),(-w-1,h) \end{matrix} \right.\right](\mathbf{E}_{2,\sigma}f)(t)dt \tag{5.7}$$

is valid, while for $\mathrm{Re}(w) < \nu h - 1$ *there holds the relation*

$$f(x) = -x^{1-(w+1)/h} \frac{d}{dx} x^{(w+1)/h}$$

$$\times \int_0^\infty H_{3,3}^{2,0} \left[xt \,\middle|\, \begin{matrix} (0,1),(-w,h) \\ (-w-1,h)(\sigma-2,2),(0,1) \end{matrix} \right] (\mathbf{E}_{2,\sigma} f)(t) dt. \tag{5.8}$$

Remark 2. Taking $\nu = 1/r$ in Theorem 1 and Theorems 2-6, we can obtain the results characterizing the theory of the Mittag-Leffler transform $\mathbf{E}_{\lambda,\sigma}$ in the spaces $L^2(\mathbf{R}_+)$ and $L^r(\mathbf{R}_+)$ $(r \geq 1)$, while from Theorem 7 we obtain the inversion of the integral transform $\mathbf{E}_{2,\sigma} f$ in $L^r(\mathbf{R}_+)$.

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