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WEISNER'S METHODIC SURVEY OF MODIFIED LAGUERRE POLYNOMIALS*

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ABSTRACT - A good number of generating functions involving modified Laguerre polynomials $L_{a,b,m,n}(x)$ have been derived by some researches ([4-7]) by suitable single interpretation to (i) the index n, to (ii) the parameter m and by suitable double interpretation to (iii) the index n and the parameter m simultaneously while applying group theoretic method of obtaining generating functions introduced by L.Weisner in the study of modified Laguerre polynomials. In this article the authors have made a modest attempt to present a comprehensive Weisner's methodic survey on the polynomials under consideration. Moreover they have shown that the results obtained by double interpretation while studying $L_{a,b,m,n}(x)$ by the application of Weisner's method can be easily derived from the results obtained by single interpretation to the index n while investigating $L_{a,b,m-n,n}(x)$, a modification of $L_{a,b,m,n}(x)$ by the same method.

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1. Introduction. In 1983, Goyal [1] defined the modified Laguerre polynomial as follows:

$$(1.1) \qquad L_{a,b,m,n}(x) = \frac{b^n(m)_n}{n!}\,{}_1F_1\left(-n, m; \frac{ax}{b}\right) \quad , \quad m \neq 0, -1, -2, \dots ,$$

satisfying the following ordinary differential equation

$$(1.2) \qquad xD_x^2u + \left(m - \frac{ax}{b}\right)D_xu + n\frac{a}{b}u = 0 \quad , \quad D_x = \frac{d}{dx} .$$

In 1955, Weisner [2] gave a method of obtaining generating functions from the Lie group view point, which is subsequently known as "Weisner's group theoretic method of obtaining generating functions" while investigating Hypergeometric polynomials.

Weisner's method of obtaining generating functions consists in constructing a partial differential equation from an ordinary differential equation satisfied by a certain special function by suitable interpretation to either the index or to the parameter of the special function under consideration and then finding a non-trivial continuous transformations

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group admitted by the partial differential equation. The above method is lucidly presented in the book "Obtaining generating functions" written by E.B.McBride [3].

Very recently this method has been extensively utilized for obtaining generating functions of modified Laguerre polynomials as defined in (1.1) by Singh and Bala [4] with the interpretation of the index n, Chongdar and Majumdar [5] by the interpretation of the parameter m, Sen and Chongdar [6] and Chongdar, Pittaluga and Sacripante [7] with the double interpretation of the index n and the parameter m simultaneously.

While investigating generating functions of $L_{a,b,m,n}(x)$ by Weisner's group theoretic method by the interpretation of the index n, Singh and Bala considered the set of operators:

$$\begin{aligned} A_1 &= y\frac{\partial}{\partial y}, \\ A_2 &= xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \\ A_3 &= bxy\frac{\partial}{\partial x} + by^2\frac{\partial}{\partial y} + (bm - ax)y \end{aligned}$$

such that

$$\begin{aligned} A_1(L_{a,b,m,n}(x)y^n) &= nL_{a,b,m,n}(x)y^n, \\ A_2(L_{a,b,m,n}(x)y^n) &= b(1-m-n)L_{a,b,m,n-1}(x)y^{n-1}, \\ A_3(L_{a,b,m,n}(x)y^n) &= (n+1)L_{a,b,m,n+1}(x)y^{n+1}. \end{aligned}$$

The following commutator relations satisfied by A_1, A_2, A_3

$$\begin{aligned} [A_1, A_2] &= -A_2, \\ [A_1, A_3] &= A_3, \\ [A_2, A_3] &= -2bA_1 - bm, \end{aligned}$$

where

$$[A, B]u = (AB - BA)u,$$

show that the set of operators $\{1, A_1, A_2, A_3\}$ generates a Lie algebra $\mathcal{L}_1$.

For obtaining generating functions by suitable interpretation of the parameter m, Chongdar and Majumdar [5] considered the set of operators

$$\begin{aligned} B_1 &= y\frac{\partial}{\partial y}, \\ B_2 &= \frac{b}{a}y\frac{\partial}{\partial x} - y, \\ B_3 &= xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - y^{-1} \end{aligned}$$

such that

$$\begin{aligned} B_1(L_{a,b,m,n}(x)y^m) &= mL_{a,b,m,n}(x)y^m \,, \\ B_2(L_{a,b,m,n}(x)y^m) &= -\, L_{a,b,m+1,n}(x)y^{m+1}, \\ B_3(L_{a,b,m,n}(x)y^m) &= (n+m+1)L_{a,b,m-1,n}(x)y^{m-1} \,. \end{aligned}$$

The following commutator relations satisfied by $B_i, i = 1, 2, 3,$

$$\begin{aligned} [B_1, B_2] &= B_2 \,, \\ [B_1, B_3] &= -\, B_3 \,, \\ [B_2, B_3] &= 1 \end{aligned}$$

show that the set of operators $\{1, B_1, B_2, B_3\}$ generates a Lie algebra $\mathcal{L}_2$.

For obtaining generating functions by suitable interpretations of the index n and the parameter m of the polynomial, Sen and Chongdar [6] considered the following operators

$$\begin{aligned} C_1 &= y\frac{\partial}{\partial y} \,, \\ C_2 &= z\frac{\partial}{\partial z} \,, \\ C_3 &= bxy^{-1}z\frac{\partial}{\partial x} + zb\frac{\partial}{\partial y} - y^{-1}z(ax+b) \,, \\ C_4 &= \frac{yz^{-1}}{a}\frac{\partial}{\partial x} \end{aligned}$$

such that

$$\begin{aligned} C_1(L_{a,b,m,n}(x)y^m z^n) &= mL_{a,b,m,n}(x)y^m z^n \,, \\ C_2(L_{a,b,m,n}(x)y^m z^n) &= nL_{a,b,m,n}(x)y^m z^n \,, \\ C_3(L_{a,b,m,n}(x)y^m z^n) &= (n+1)L_{a,b,m-1,n+1}(x)y^{m-1}z^{n+1} \,, \\ C_4(L_{a,b,m,n}(x)y^m z^n) &= -\, L_{a,b,m+1,n-1}(x)y^{m+1}z^{n-1} \,. \end{aligned}$$

The following commutator relations satisfied by $C_i, i = 1, 2, 3, 4$

$$\begin{cases} [C_i, C_j] = 0 & \text{when} \quad i = 1 \quad ; j = 2 \\ \quad\quad\;\; = (-1)^{i+j+1}C_j & \quad\quad\quad\; i = 1, 2 \quad ; j = 3, 4 \\ \quad\quad\;\; = 1 & \quad\quad\quad\; i = 3 \quad ; j = 4 \end{cases}$$

show that the set of operators $\{1, C_i, i = 1, 2, 3, 4\}$ generates a Lie algebra $\mathcal{L}_3$.

Finally while investigating generating functions of the polynomial under consideration with the suitable interpretations of the index n and the parameter m of the polynomial

Chongdar, Pittaluga and Sacripante considered the following operators

$$
\begin{aligned}
D_1 &= y\frac{\partial}{\partial y}\,, \\
D_2 &= z\frac{\partial}{\partial z}\,, \\
D_3 &= \frac{b}{a}y\frac{\partial}{\partial x} - y\,, \\
D_4 &= xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - y^{-1}\,, \\
D_5 &= bxy^{-1}z\frac{\partial}{\partial x} + zb\frac{\partial}{\partial y} - y^{-1}z(ax+b)\,, \\
D_6 &= \frac{yz^{-1}}{a}\frac{\partial}{\partial x}
\end{aligned}
$$

such that

$$
\begin{aligned}
D_1(L_{a,b,m,n}(x)y^m z^n) &= mL_{a,b,m,n}(x)y^m z^n\,, \\
D_2(L_{a,b,m,n}(x)y^m z^n) &= nL_{a,b,m,n}(x)y^m z^n\,, \\
D_3(L_{a,b,m,n}(x)y^m z^n) &= -\,L_{a,b,m+1,n}(x)y^{m+1} z^n\,, \\
D_4(L_{a,b,m,n}(x)y^m z^n) &= (n+m+1)L_{a,b,m-1,n}(x)y^{m-1} z^n\,, \\
D_5(L_{a,b,m,n}(x)y^m z^n) &= (n+1)L_{a,b,m-1,n+1}(x)y^{m-1} z^{n+1}\,, \\
D_6(L_{a,b,m,n}(x)y^m z^n) &= -\,L_{a,b,m+1,n-1}(x)y^{m+1} z^{n-1}\,.
\end{aligned}
$$

The following commutator relations satisfied by $D_i, i = 1, 2, \ldots, 6$

$$
\begin{cases}
[D_i, D_j] = 0 & \text{when } i = 1 \;;\; j = 2 \\
 & \phantom{\text{when }} i = 2 \;;\; j = 3,4 \\
 & \phantom{\text{when }} i = 3,4 \;;\; j = 5,6 \\
 = (-1)^{i+j}D_j & \phantom{\text{when }} i = 1 \;;\; j = 3,4 \\
 = (-1)^{i+j+1}D_j & \phantom{\text{when }} i = 1,2 \;;\; j = 5,6 \\
 = 1 & \phantom{\text{when }} i = 3 \;;\; j = 4 \\
 & \phantom{\text{when }} i = 5 \;;\; j = 6
\end{cases}
$$

show that the set of operators $\{1, D_i, i = 1, 2 \ldots, 6\}$generates a Lie algebra $\mathcal{L}_4$ and each of the sub sets $\{1, D_i, i = 1, 3, 4\}$ and $\{1, D_i, i = 1, 2, 5, 6\}$ generates a sub algebra of $\mathcal{L}_4$.

The object of the present article is to make a comprehensive study on $L_{a,b,m-n,n}(x)$ - a modification of $L_{a,b,m,n}(x)$ - for obtaining generating functions by the application of Weisner's group theoretic method with the interpretation of the index n of the polynomial under consideration and to make a review on the previous works.

In fact, in the present discussion it has been pointed out that the results obtained by Sen and Chongdar [6] with the help of double interpretation during the application of Weisner's group theoretic method of obtaining generating functions on the modified Laguerre polynomials $L_{a,b,m,n}(x)$, may be derived as the particular cases of the results derived here.
We also like to point it out that our results (2.17) - (2.18) together with the results due to Chongdar and Majumder [5] give rise to the main generating relation of the present authors [7] obtained by double interpretations to n and m, the index and the parameter of $L_{a,b,m,n}(x)$.

2. Study of $L_{a,b,m-n,n}(x)$. The differential equation satisfied by $L_{a,b,m-n,n}(x)$ is given by

$$xD_x^2u+\left(m-n-\frac{ax}{b}\right)D_xu+\frac{a}{b}nu=0 \quad , \quad D_x=\frac{d}{dx}\,. \tag{2.1}$$

In this Section, some generationg functions of $L_{a,b,m-n,n}(x)$ have been derived by using Weisner's group theoretic method with the suitable interpretation of the index n.

i) Group theoretic discussion.

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$, u by $\nu(x,y)$ in (2.1), we get the following partial differential equation:

$$x\frac{\partial^2\nu}{\partial x^2}-y\frac{\partial^2\nu}{\partial x\partial y}+\left(m-\frac{ax}{b}\right)\frac{\partial\nu}{\partial x}+\frac{ay}{b}\frac{\partial\nu}{\partial y}=0\,. \tag{2.2}$$

Thus $\nu_1(x,y)=L_{a,b,m-n,n}(x)y^n$ is a solution of (2.2) since $L_{a,b,m-n,n}(x)$ is a solution of (2.1).

We now define the infinitesimal operators A, B, C as follows

$$\begin{aligned} A&=y\frac{\partial}{\partial y}\,,\\ B&=bxy\frac{\partial}{\partial x}-by^2\frac{\partial}{\partial y}-[ax+(1-m)b]y\,,\\ C&=\frac{y^{-1}}{a}\frac{\partial}{\partial x}\,, \end{aligned} \tag{2.3}$$

such that

$$\begin{aligned} A(L_{a,b,m-n,n}(x)y^n)&=nL_{a,b,m-n,n}(x)y^n\,,\\ B(L_{a,b,m-n,n}(x)y^n)&=(n+1)L_{a,b,m-n-1,n+1}(x)y^{n+1}\,,\\ C(L_{a,b,m-n,n}(x)y^n)&=-\,L_{a,b,m-n+1,n-1}(x)y^{n-1}\,. \end{aligned} \tag{2.4}$$

The commutator relations satisfied by A, B, C are

$$[A, B] = B \quad , \quad [A, C] = -C \quad , \quad [B, C] = 1 \,. \tag{2.5}$$

The above commutator relations show that the set of operators $\{1, A, B, C\}$ generates a Lie algebra $\mathcal{L}_5$.

It can be easily shown that the partial differential operator

$$L = x\frac{\partial^2}{\partial x^2} - y\frac{\partial^2}{\partial x \partial y} + \left(m - \frac{ax}{b}\right)\frac{\partial}{\partial x} + \frac{ay}{b}\frac{\partial}{\partial y} \,,$$

which can be expressed as

$$\frac{b}{a}L = BC + A + 1 \,, \tag{2.6}$$

commutes with each A, B, C i.e.

$$\left[\frac{b}{a}L, A\right] = \left[\frac{b}{a}L, B\right] = \left[\frac{b}{a}L, C\right] = 0 \,. \tag{2.7}$$

The extended form of the groups generated by A, B, C are as follows

$$\begin{cases} e^{a_1 A} f(x,y) = f(x, e^{a_1} y) \,, \\ e^{a_2 B} f(x,y) = (1 + a_2 by)^{m-1} \exp(-a a_2 xy) f\left(x(1 + a_2 by), \dfrac{y}{1 + a_2 by}\right) \,, \\ e^{a_3 C} f(x,y) = f\left(x + \dfrac{a_3}{a} y^{-1}, y\right) \,. \end{cases} \tag{2.8}$$

Thus we get

$$\begin{gathered} e^{a_3 C} e^{a_2 B} e^{a_1 A} f(x,y) = \\ (1 + a_2 by)^{m-1} \exp\left[-(axy + a_3)a_2\right] \cdot \\ \cdot f\left((x + \frac{a_3}{a} y^{-1})(1 + a_2 by), \frac{y}{1 + a_2 by}\right) \,. \end{gathered} \tag{2.9}$$

ii) Generating functions.

From (2.2) we see that $\nu(x,y) = L_{a,b,m-n,n}(x)y^n$ is a solution of the system

$$\begin{cases} L\nu = 0 \\ (A - n)\nu = 0 \,. \end{cases} \tag{2.10}$$

It can be easily verified that

$$S \cdot \frac{b}{a} L(L_{a,b,m-n,n}(x)y^n) = \frac{b}{a} L \cdot S(L_{a,b,m-n,n}(x)y^n) = 0\,,$$

where

$$S = e^{a_3 C} e^{a_2 B} e^{a_1 A}\,.$$

Thus the transformation $S(L_{a,b,m-n,n}(x)y^n)$ is annihilated by $\frac{b}{a}L$.

Putting $a_1 = 0$ and writing $f(x,y) = L_{a,b,m-n,n}(x)y^n$ in (2.9), we get

$$\text{(2.11)} \qquad e^{a_3 C} e^{a_2 B}(L_{a,b,m-n,n}(x)y^n) =$$
$$= (1 + a_2 by)^{m-n-1} \exp\left[-a_2(axy + a_3)\right] \cdot$$
$$\cdot L_{a,b,m-n,n}\left((x + \frac{a_3}{a} y^{-1})(1 + a_2 by)\right)\,.$$

But,

$$\text{(2.12)} \qquad e^{a_3 C} e^{a_2 B}(L_{a,b,m-n,n}(x)y^n) =$$
$$= y^n \sum_{p=0}^{n+k} \frac{(-a_3/y)^p}{p!} \sum_{k=0}^{\infty} \frac{(a_2 y)^k}{k!} (n+1)_k L_{a,b,m-n-k+p,n+k-p}(x)\,.$$

Equating (2.11) and (2.12) we get

$$\text{(2.13)} \qquad (1 + a_2 by)^{m-n-1} \exp[-(axy + a_3)a_2] \cdot$$
$$\cdot L_{a,b,m-n,n}(x)\left((x + \frac{a_3}{a} y^{-1})(1 + a_2 by)\right) =$$
$$= \sum_{p=0}^{n+k} \frac{(-a_3/y)^p}{p!} \sum_{k=0}^{\infty} \frac{(a_2 y)^k}{k!} (n+1)_k L_{a,b,m-n-k+p,n+k-p}(x)\,.$$

We now discuss the following particular cases of the above generating relation(2.13).

Case 1 : putting $a_3 = 0$ and then replacing $a_2 y$ by t in (2.13) we get

$$\text{(2.14)} \qquad (1 + bt)^{m-n-1} \exp(-axt) L_{a,b,m-n,n}(x)(x(1 + bt)) =$$
$$= \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} L_{a,b,m-n-k,n+k}(x) t^k\,.$$

Case 2 : putting $a_2 = 0$ and then replacing $-a_3/y$ by t in (2.13), we get

$$L_{a,b,m-n,n}(x)\left(x - \frac{t}{a}\right) = \tag{2.15}$$

$$= \sum_{p=0}^{n} \frac{1}{p!} L_{a,b,m+p-n,n-p}(x) t^p .$$

Case 3 : taking $a_2 a_3 \neq 0$ without any loss of generality we can choose $a_2 y = t_1$ and $-a_3/y = t_2$ in (2.13) and then we get

$$(1+bt_1)^{m-n-1}\exp[-(ax - t_2)t_1] L_{a,b,m-n,n}\left(\left(x - \frac{t_2}{a}\right)(1+bt)\right) = \tag{2.16}$$

$$= \sum_{p=0}^{n+k} \frac{{t_2}^p}{p!} \sum_{k=0}^{\infty} \frac{{t_1}^k}{k!} (n+1)_k L_{a,b,m-n-k+p,n+k-p}(x) .$$

Now if we replace m by $m+n$ on both sides of (2.14)-(2.16), we get the results found derived in [6].

Now we proceed to derive the main result of the present authors [(3.3) in [7]] by making use of our results (2.14)-(2.15) together with the results of Congdar and Majumdar. In fact, we see the right hand side of (3.3) in [7], with the help of the relations (Case 2, of[5]), (Case 1 of [5]), (2.15) and (2.14), is

$$\sum_{r=0}^{\infty} \frac{(a_5 z/y)^r}{r!} \sum_{s=0}^{n+r} \frac{(-a_6 y/z)^s}{s!} \sum_{p=0}^{\infty} \frac{(-a_3 y)^p}{p!} \sum_{k=0}^{\infty} \frac{(-a_4/y)^k}{k!} \cdot$$

$$\cdot (n+1)_r (-n-p-m+1)_k L_{a,b,m+p-k-r+s,n+r-s}(x) =$$

$$= \left(1 + \frac{a_4}{y}\right)^{m-1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{a_5 z}{y(1+\frac{a_4}{y})}\right)^r \sum_{s=0}^{n+r} \frac{1}{s!} \left(-\frac{a_6 y}{z}(1+\frac{a_4}{y})\right)^s \cdot$$

$$\cdot \sum_{p=0}^{\infty} \frac{1}{p!} \left(-a_3 y(1+\frac{a_4}{y})\right)^p (n+1)_r L_{a,b,m+p-r+s,n+r-s}\left(x(1+\frac{a_4}{y})\right) =$$

$$= (1+\frac{a_4}{y})^{m-1} \exp\left[-a_3 y(1+\frac{a_4}{y})\right] \cdot$$

$$\cdot \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{a_5 z}{y(1+\frac{a_4}{y})}\right)^r \sum_{s=0}^{n+r} \frac{1}{s!} \left(-\frac{a_6 y}{z}(1+\frac{a_4}{y})\right)^s (n+1)_r \cdot$$

$$\cdot L_{a,b,m-r+s,n+r-s}\left((1+\frac{a_4}{y})(x+\frac{b}{a}a_3 y)\right) =$$

$$= (1+\frac{a_4}{y})^{m-1}\exp\left[-a_3y(1+\frac{a_4}{y})\right]\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{a_5z}{y(1+\frac{a_4}{y})}\right)^r\cdot$$

$$\cdot(n+1)_r L_{a,b,m-r,n+r}\left((1+\frac{a_4}{y})(x+\frac{b}{a}a_3y+\frac{a_6y}{za})\right) =$$

$$= (1+\frac{a_4}{y}+\frac{ba_5z}{y})^{m-1}\exp\left[-a_3y(1+\frac{a_4}{y}+\frac{ba_5z}{y})-\frac{aa_5z}{y}(x+\frac{a_6y}{az})\right]\cdot$$

$$\cdot L_{a,b,m,n}\left((x+\frac{b}{a}a_3y+\frac{a_6y}{za})(1+\frac{a_4}{y}+\frac{ba_5z}{y})\right),$$

which is the left hand side of (3.3)of the result found derived in [7].

Other variants of (3.3) in [7] can be derived by using the four generating relations, two of Chongdar and Majundar and two of this Section.

It may be noted that the importance of the result (3.3) in [7] lies in the fact that whenever one knows the sum, in the closed form, of a quadruple generating series like the right member of (3.3) in [7], one can verify the same by classical method, but prior to the existence of a result like (3.3) in [7] nobody could even guess such a generating relation without the help of group theoretic method.

iii) Relation of $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_5$.

It may be noted that the commutator relations satisfied by the operators generating $\mathcal{L}_2$ can be compared with the following commutator relations of $\mathcal{L}_4$:

$$[D_1, D_3] = D_3 \quad , \quad [D_1, D_4] = -D_4 \quad , \quad [D_3, D_4] = 1 .$$

It, therefore, follows that the Lie algebra $\mathcal{L}_2$ is isomorphic with the sub-Lie algebra generated by D_1, D_3, D_4 .

Again the commutator relations satisfied by the operators of the Lie algebra $\mathcal{L}_5$ can be compared with the folliwing commutator relations of $\mathcal{L}_4$:

$$[D_2, D_5] = D_5 \quad , \quad [D_2, D_6] = -D_6 \quad , \quad [D_5, D_6] = 1 .$$

It,therefore, follows that the Lie algebra $\mathcal{L}_5$ is also isomorphic with the sub-Lie algebra generated by D_2, D_5, D_6 .
Therefore we can state that $\mathcal{L}_4 = \mathcal{L}_2 \oplus \mathcal{L}_5$.

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