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SOME FUNCTIONS RELATED TO CONVEX TOPOLOGICAL SPACES

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Abstract :The aim of this paper is to introduce a new type of convex topological space called semi-compatible with respect to the topology on a given set and the convexity as introduced by Van.DeVel. Further new types of functions has been introduced using both the topology and the convexity;the mutual relations of such mapping have also been studied.

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1 Introduction

The development of 'abstract convexity' has emanated from different sources in different ways; the first type of development basically banked on generalization of particular problems such as separation of convex sets [3], extremality [4]; [2], or continuous selection [9]. The second type of development lay before the reader such axiomatizations, which, in every case of design, express of particular point of view of convexity. With the view point of generalized topology which enters into convexity via the closure or hull operator, Schmidt[1953] and Hammer[1955], [1963], [1963b] introduced some axioms to explain abstract convexity. The arising of convexity from algebraic operations and the related property of domainfitness receive attentions in Birchoff and Frink[1948] Schmidt[1953] and Hammer[1963].

Throughout this paper the axiomatizations as proposed by M. L. J. Van De Vel in his papers in the seventies and finally incorporated in Theory of Convex Structure [11] will be followed.

In [12] the author has discussed 'Topology and Convexity on the same set' and introduced the compatibility of the topology with a convexity on the same underlying set. At the very early stage of this paper we have set aside the concept of compatibility and started just with a triplet(X,$\tau, \mathcal{C}$)and called it convex topological space only to bring back'compatibility'in another way subsequently. With his compatibility, however, VanDevel has called the triplet (X,$\tau, \mathcal{C}$)a topological convex structure.

It is however seen that in many cases where compatibility is expected our definition serves the purpose.

In this paper, Art 2 deals with some early definitions and Art 3 envisages a new topology generated on a convex topological spaces via convexity; by (X,$\tau, \mathcal{C}$)is a convex topological space (CTS in short) then the generated topology τ_*, in general, is such that $\tau_* \subseteq \tau$; it has been shown by an example that $\tau_* \subset \tau$. In Art 4, some new types of functions have been introduced and in Art 5, we have introduced a special type of convex topological space. The last Art 6 deals with some basic properties of $\theta - \mathcal{C}$ somewhat open function.

2 Prerequisites

Definition 2.1 [12] Let X be a nonempty set. A family $\mathcal{C}$ of subsets of the set X is called a convexity on X if

1. $\Phi, X \in \mathcal{C}$

2. $\mathcal{C}$ is stable for intersection, i. e. if $\mathcal{D} \subseteq \mathcal{C}$ is nonempty then $\cap\mathcal{D} \in \mathcal{C}$.

3. $\mathcal{C}$ is stable for nested unions, i. e. if $\mathcal{D} \subseteq \mathcal{C}$ is nonempty and totally ordered by set inclusion then $\cup\mathcal{D} \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convex structure. The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets.

Definition 2.2 [12] Let $\mathcal{C}$ be a convexity on a set X. Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by

$co(A) = \cap\{C : A \subseteq C \in \mathcal{C}\}$.

Note 2.3 [12] Let $(X, \mathcal{C})$ be a convex structure and let Y be a subset of X. The family of sets $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a convexity on Y; it is called the relative convexity of Y.

Note 2.4 [12] The hull operator co_Y of a subspace $(Y, \mathcal{C}_Y)$ satisfies the following : $\forall A \subseteq Y : co_Y(A) = co(A) \cap Y$.

Definition 2.5 Let (X, τ) be a topological space. Let $\mathcal{C}$ be a convexity on X. Then the triplet $(X, \tau, \mathcal{C})$ is called a convex topological space (CTS, in short).

3 New Topology τ_* on CTS

In this article we have introduced a new topology τ_* on CTS with some examples on such topologies.

Let $(X, \tau, \mathcal{C})$be a CTS. Let $A \subseteq X$. We consider the set A_*, where A_* is defined as follows : $A_* = \{x \in X : co(U) \cap A \neq \Phi, x \in U \in \tau\}$.
From the definition of A_*, it is clear that $A \subseteq \overline{A} \subseteq A_*$.

Proposition 3.1 Let $(X, \tau, \mathcal{C})$ be a CTS; then

1. $\Phi_* = \Phi, X_* = X$.
2. $A, B \subseteq X and A \subseteq B\ then A_* \subseteq B_*$.
3. $(A \cup B)_* = A_* \cup B_*$.

Proof:1. Obvious.

2. Obvious.

3. $A \subseteq (A \cup B) \Rightarrow A_* \subseteq (A \cup B)_*$ and $B \subseteq (A \cup B) \Rightarrow B_* \subseteq (A \cup B)_*$. So $A_* \cup B_* \subset (A \cup B)_*$. To prove the converse part, let $x \in (A \cup B)_*$. If possible let $x \notin A_*$ and also $x \notin B_*$. Then $\exists$ U_1 and U_2 such that $co(U_1) \cap A = \Phi$ and $co(U_2) \cap B = \Phi$ where $x \in U_1 \in \tau$ and $x \in U_2 \in \tau$. Now $x \in U_1 \cap U_2 \in \tau$ and $co(U_1 \cap U_2) \subseteq co(U_1)$ also $co(U_1 \cap U_2) \subseteq co(U_2)$.
Hence $co(U_1 \cap U_2) \cap (A \cup B) = (co(U_1 \cap U_2) \cap A) \cup (co(U_1 \cap U_2) \cap B) = \Phi$-which contradicts the fact that $x \in (A \cup B)_*$. Hence either $x \in A_*$ or $x \in B_*$, i. e. $(A \cup B)_* \subseteq A_* \cup B_*$. Consequently $(A \cup B)_* = A_* \cup B_*$.

Theorem 3.2 Let us consider the collection $\tau_* = \{A^c : A \subseteq X \text{and} A = A_*\}$. Then τ_* is a topology on X such that $\tau_* \subseteq \tau$.

Proof : 1. $\phi_* = \phi$ and $X_* = X$, so $\phi, X \in \tau_*$.

2. Let $A_1, A_2, - - - A_n \in \tau_*$ and $B = \bigcap_{i=1}^{n} A_i$.

Now $(B^c)_* = [(\bigcap_{i=1}^{n} A_i)^c]_*$

$= [\bigcup_{i=1}^{n} A_i^c]_* = \bigcup_{i=1}^{n} (A_i^c)_*$[by Proposition 3.1]

$= \bigcup_{i=1}^{n} A_i^c$ [since $A_i \in \tau_*$, $(A_i^c)_* = A_i^c$]

$= (\bigcap A_i)^c = B^c$.

Thus $B = (B^c)^c \in \tau_*$.

3. Let $A_\alpha (\alpha \in \Lambda) \in \tau_*$. Let $B = \bigcup_{\alpha \in \Lambda} A_\alpha$.

First we prove that $(\bigcap_{\alpha \in \Lambda} A_\alpha^c)_* = \bigcap_{\alpha \in \Lambda} A_\alpha^c$.

Now $\bigcap_{\alpha \in \Lambda} A_\alpha^c \subseteq (\bigcap_{\alpha \in \Lambda} A_\alpha^c)_*$.

Again $\bigcap_{\alpha \in \Lambda} A_\alpha^c \subseteq A_\alpha^c$; $\forall \alpha \in \Lambda$.

$\Rightarrow (\bigcap_{\alpha \in \Lambda} A_\alpha^c)_* \subseteq (A_\alpha^c)_* = A_\alpha^c$ [since $A_\alpha \in \tau_*$] $\forall \alpha \in \Lambda$.

$\Rightarrow (\bigcap_{\alpha \in \Lambda} A_\alpha^c)_* \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha^c$.

Hence $(\bigcap_{\alpha \in \Lambda} A_\alpha^c)_* = \bigcap_{\alpha \in \Lambda} A_\alpha^c$. Now, $(B^c)_* = ((\bigcup_{\alpha \in \Lambda} A_\alpha)^c)_* = \bigcap_{\alpha \in \Lambda} A_\alpha^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c = (\bigcup_{\alpha \in \Lambda} A_\alpha)^c = B^c$ and so $B = (B^c)^c \in \tau_*$. Thus τ_* is a topology on X.

Let $A \in \tau_*$ and let $B = A^c$. Then we have $B = B_*$. Since for any $B \subseteq X$, $B \subseteq \overline{B} \subseteq B_*$, we have$B = \overline{B} = B_*$. Thus $B^c = A \in \tau$. Hence τ_* is a topology on X such that $\tau_* \subseteq \tau$.

Note 3.3 The members of τ_* are called convex-open sets and a set $A \subseteq X$is called convex-closed if $A^c \in \tau_*$.

In the following examples we will show that τ_* may be strictly coarser than τ.

Example 3.4 Let $X = [-1, 1]$. Let a topology τ on X consists of those subsets of X which either do not contain $\{0\}$ or contain $(-1, 1)$.

Let a convexity $\mathcal{C}$ on X be defined as follows:

$\mathcal{C} = \{X, \Phi\} \cup \{\{x\} : x \in X\}$.

It is clear that, $\{1\}, \{-1\}, \{-1, 1\}$ and any set containing $\{0\}$ are τ closed sets. Now $0 \in \{1\}_*$, $0 \in \{-1\}_*$ and $0 \in \{-1, 1\}_*$ so that $\{1\}_* \neq \{1\}$,

$\{-1\}_* \neq \{-1\}$,

$\{-1,1\}_* \neq \{-1,1\}$. Again let A be any subset such that $0 \in A$. Let $p \notin A$. Then $\{p\} \neq \{0\}$. Since $\{p\} \in \tau$ as well as $\{p\} \in \mathcal{C}$, we claim that $p \notin A_*$. Hence $A_* = A$. Hence the new topology τ_* consists of all those subsets which do not contain $\{0\}$ together with X. Thus $\{\Phi, X\} \subset \tau_* \subset \tau$.

Example 3.5 Let $X = \{a, b, c\}$.
$\tau = \{\Phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.
$\mathcal{C} = \{\Phi, X, \{a\}, \{b\}\}$.
Here, $\tau_* = \{\Phi, X, \{b\}, \{a\}, \{a, b\}\}$. Hence, $\tau_* \subset \tau$.
In CTS, we see that the topology τ_* is coarser than the original topology and the above examples show that τ_* may be strictly coarser than τ. So it is natural to ask whether, for an arbitrary set X, there exists a topology and a convexity on X for which τ_* coincides with the original topology. The following examples serve the purpose.

Example 3.6 Let X be any set and $\tau = \mathcal{P}(X)$. Let $\mathcal{C}$ be consists of Φ, X and all singletons. Here for any set $A \subseteq X$, we have $A = A_*$. Hence in this case, we have $\tau_* = \tau = \mathcal{P}(X)$.

Again for any set X, if $\tau = \{\Phi, X\}$ and $\mathcal{C}$ be any convexity on X, then $\tau_* = \{X, \Phi\} = \tau$. These are the trivial examples for which the topology τ_* coincide with the original topology. We now give another non trivial interesting example.

Example 3.7 Let X be a set and $\mathcal{C}$ be any convexity on X. From the definition of convexity $\mathcal{C}$ on X, it is clear that $\mathcal{C}$ is a base for some topology. Let this topology be denoted by $\tau(\mathcal{C})$. Now let us consider the convex topological space $(X, \tau(\mathcal{C}), \mathcal{C})$. Since $\tau_* \subseteq \tau(\mathcal{C})$, we only show that every $\tau(\mathcal{C})$ closed sets are also τ_* closed. Let A be $\tau(\mathcal{C})$ closed, i.e. $A = \overline{A}$. To prove $A = A_*$, it is sufficient to show that $A_* \subseteq \overline{A}$[since $A = \overline{A} \subseteq A_*$]. Let $x \in A_*$ and $x \in U \in \tau(\mathcal{C})$. If $U \in \mathcal{C}$, then $co(U) \cap A = U \cap A \neq \phi$[since $x \in A_*$]. Again if $U \in \tau(\mathcal{C}) \setminus \mathcal{C}$, then $\exists C \in \mathcal{C}$ such that $x \in C \subseteq U$. Now $C \cap A = co(C) \cap A \neq \phi$[since $x \in A_*$]$\Rightarrow U \cap A \neq \phi$. Thus $x \in \overline{A}$ and we get $A = \overline{A} = A_*$. Consequently $\tau_* = \tau(\mathcal{C})$.

4 $\theta - \mathcal{C}$ Somewhat Open Functions

In this article we define new types of functions on CTS with examples. Also we discuss their mutual relationship.

Definition 4.1 [5] Let (X,σ) and (Y,τ) be topological spaces. A function $f:(X,\sigma)\rightarrow(Y,\tau)$ is said to somewhat open provided if $U(\neq\Phi)\in\sigma$, then $\exists$ a $V\in\tau$ such that$V\neq\Phi$ and $V\subseteq f(U)$.

Definition 4.2 A function $f:(X,\tau,\mathcal{C}_1)\longrightarrow(Y,\nu,\mathcal{C}_2)$ is said to be θ-$\mathcal{C}$-open if for each $x\in X$ and each nbd. U of x, $\exists$ a nbd.V of $f(x)$ in Y such that $V_*\subseteq f(U_*)$.

Definition 4.3 A function $f:(X,\tau,\mathcal{C}_1)\longrightarrow(Y,\nu,\mathcal{C}_2)$ is said to be θ-$\mathcal{C}$-somewhat open [respectively almost somewhat $\mathcal{C}$-open, weak somewhat $\mathcal{C}$-open] (briefly θ-$\mathcal{C}$. sw. o, a. sw. $\mathcal{C}$. o, w. sw. $\mathcal{C}$. o respectively) if $U\in\tau$ and $U\neq\Phi$ then $\exists$ a $V\in\nu$ such that $V\neq\Phi$ and $V_*\subseteq f(U_*)$ [respectively $V\subseteq f(\mathrm{int}(U_*))$, $V\subseteq f(U_*)$]

Remark 4.4 We obtain the following diagram from the definitions.

DIAGRAM - I

$$\begin{array}{ccccc}
\text{open } f^{\underline{n}}_{..} & \Longrightarrow & \text{somewhat open } f^{\underline{n}}_{..} & \Longrightarrow & \text{almost somewhat } \mathcal{C}\text{-open } f^{\underline{n}}_{..} \\
\not\Uparrow\not\Downarrow & & \not\Uparrow\not\Downarrow & & \not\Uparrow\Downarrow \\
\theta\text{-}\mathcal{C}\text{-open } f^{\underline{n}}_{..} & \Longrightarrow & \theta\text{-}\mathcal{C}\text{-somewhat open } f^{\underline{n}}_{..} & \Longrightarrow & \text{weak somewhat } \mathcal{C}\text{-open } f^{\underline{n}}_{..}
\end{array}$$

Remark 4.5 The following examples enable us to realize that none of these implications is reversible.

Example 4.6 Let $X=\{a,b,c\}$, $\tau=\{\Phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}$, $\mathcal{C}_1=\{\Phi,X,\{a\},\{c\}\}$, $\upsilon=\{\Phi,X,\{a,b\},\{c\}\}$, $\mathcal{C}_2=\{\Phi,X,\{a,b\},\{c\}\}$ and the function $f:(X,\tau,\mathcal{C}_1)\rightarrow(X,\upsilon,\mathcal{C}_2)$ be defined as follows : $f(a)=a$, $f(b)=b$, $f(c)=c$. Here f is $\theta-\mathcal{C}$-somewhat open function. Now consider the point b in $(X,\tau,\mathcal{C}_1)$. $\{b,c\}$ is a nbd.of b in $(X,\tau,\mathcal{C}_1)$. But in $(X,\upsilon,\mathcal{C}_2)$, $X_*=X$ and $\{a,b\}_*=\{a,b\}$ and $\{a,b\}_*\nsubseteq f(\{b,c\})$, $X_*\nsubseteq f(\{b,c\})$. Hence f is not $\theta-\mathcal{C}$-open function.

Example 4.7 Let $X=\{a,b,c\},\tau=\{\Phi,X,\{b\},\{c\},\{b,c\}\},\mathcal{C}_1=\{\Phi,X,\{a\}\},\upsilon=\{\Phi,X,\{c\},\{$ $\mathcal{C}_2=\{\Phi,X,\{c\}\}$and the function $f:(X,\tau,\mathcal{C}_1)\rightarrow(X,\upsilon,\mathcal{C}_2)$ be defined as follows : $f(a)=c$, $f(b)=a$, $f(c)=b$.

Here f is $\theta-\mathcal{C}$-open, $\theta-\mathcal{C}$-somewhat open function, but f is not open and not somewhat open function.

For the convex topological space $(X, \tau, \mathcal{C}_1)$, $\{b\}_* = X$, $\{c\}_* = X$, $\{b, c\}_* = X$. Thus for any $U \in \tau(\neq \Phi)$, we take $V = X$ in $(X, \upsilon, \mathcal{C}_2)$ such that $V_* \subseteq f(U_*)$. So f is $\theta - \mathcal{C}$-somewhat open function.

Since for any $U \in \tau(\neq \Phi)$, $U_* = X$, we can say that f is $\theta - \mathcal{C}$-open function.

Here $\{c\} \in \tau$ but $f(\{c\}) = \{b\} \notin \upsilon$. So f is not open function. For $\{b\} \in \tau$, since $\exists$ a nonempty $V \in \upsilon$ such that $V \nsubseteq f(\{b\}) = \{a\}$, f is not somewhat open function.

Example 4.8 Let $X = \{a, b\}$, $\tau = \{\Phi, X, \{a\}, \{b\}\}$, $\mathcal{C}_1 = \{\Phi, X, \{a\}\}$, $Y = \{x, y, z\}$, $\upsilon = \{\Phi, Y, \{x\}, \{y\}, \{x, y\}\}$, $\mathcal{C}_2 = \{\Phi, Y\}$, and the function $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$ be defined as follows : $f(a) = x$, $f(b) = y$. Here f is open but not $\theta - \mathcal{C}$ open function. Clearly f is an open function. In the convex topological space $(X, \tau, \mathcal{C}_1)$, $\{b\}_* = \{b\}$.

In the convex topological space $(Y, \upsilon, \mathcal{C}_2)$, $\{x\}_* = Y$, $\{y\}_* = Y$, $\{x, y\}_* = Y$.

So if we take $U = \{b\} \in \tau$ then there is no $V \in \upsilon$ such that $V_* \subseteq f(U_*)$.

Hence f is not $\theta - \mathcal{C}$-somewhat open function. Consequently f is not $\theta - \mathcal{C}$-open function.

Example 4.9 Let $X = \{a, b\}$, $\tau = \{\Phi, X, \{a\}\}$, $\mathcal{C}_1 = \{\Phi, X\}$, $\upsilon = \{\Phi, X\}$, $\mathcal{C}_2 = \{\Phi, X\}$and the function $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$ be defined as follows : $f(a) = b$, $f(b) = a$.

Here f is almost somewhat $\mathcal{C}$-open but not somewhat open function.

In the convex topological space $(X, \tau, \mathcal{C}_1)$, $X_* = X$ and $int(X_*) = int(X) = X$, $\{a\}_* = X$ and $int(\{a\}_*) = int(X) = X$. Hence clearly f is almost somewhat $\mathcal{C}$-open function (take $V = X$).

Here $\{a\} \in \tau$ but $X \nsubseteq \{b\}$, i.e., $X \nsubseteq f(\{a\})$. Hence f is not somewhat open function.

Example 4.10 Let $X = \{a, b, c\}$, $\tau = \{\Phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{C}_1 = \{\Phi, X, \{a\}\}$, $\upsilon = \{\Phi, X, \{c\}\}, \mathcal{C}_2 = \{\Phi, X\}$ and the function $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$ be defined as follows:$f(a) = a$, $f(b) = b$, $f(c) = c$.

Here f is weak somewhat $\mathcal{C}$-open but not $\theta - \mathcal{C}$ somewhat open and not almost somewhat $\mathcal{C}$-open function.

In the convex topological space $(X, \tau, \mathcal{C}_1)$, $\{a\}_* = X$, $\{b\}_* = \{b, c\}$, $\{a, b\}_* = X$. If $U = \{b\}$ then take $V = \{c\} \in \upsilon$ such that $V \subseteq f(\{b\}_*)$.

Again if $U = \{a\}$ or, $U = \{a, b\}$, then take $V = X$. So f is a weak somewhat $\mathcal{C}$-open function.

Now in the convex topological space $(X, \tau, \mathcal{C}_2)$, $\{c\}_* = X$ and $X_* = X$. Thus for $U = \{b\} \in \tau$, there is no $V \in \upsilon$ such that $V_* \subseteq f(U_*)[f(U_*) = f(\{b_*\}) = f(\{b,c\}) = \{b,c\}]$.

Hence f is not $\theta - \mathcal{C}$ somewhat open function.

Again for $U = \{b\} \in \tau$, $\{b\}_* = \{b,c\}$ and $int(U_*) = int(\{b\}_*) = int(\{b,c\} = \{b\}$. But there is no $V \in \upsilon$ such that $V \subseteq f(int(U_*))$.

Hence f is not almost somewhat $\mathcal{C}$-open function.

Example 4.11 Let $X = \{a,b\}$, $\tau = \{\Phi, X, \{a\}\}$, $\mathcal{C}_1 = \{\Phi, X\}$, $\upsilon = \{\Phi, X\}$, $\mathcal{C}_2 = \{\Phi, X\}$ and the function $f : (X, \tau, \mathcal{C}_1) \to (X, \upsilon, \mathcal{C}_2)$ be defined as follows : $f(a) = a$, $f(b) = b$.

Here f is almost somewhat $\mathcal{C}$-open function but not somewhat open function.

In the convex topological space $(X, \tau, \mathcal{C}_1)$, $\{a\}_* = X$. So, $int(\{a\}_*) = int(X) = X$. Thus for $U = \{a\} \in \tau$, take $V = X \in \upsilon$ such that $V \subseteq f(int(U_*))$. Hence f is almost somewhat $\mathcal{C}$ open function.

Again for $U = \{a\} \in \tau$, there is no $V \in \upsilon$ such that $V \subseteq f(U)$. Hence f is not somewhat open function.

Example 4.12 Let $X = \{a,b,c\}$, $\tau = \{\Phi, X, \{a\}, \{b,c\}\}$, $\mathcal{C}_1 = \{\Phi, X, \{a\}\}$, $\upsilon = \{\Phi, X, \{a\}, \{b,c\}\}$, $\mathcal{C}_2 = \{\Phi, X\}$ and the function $f : (X, \tau, \mathcal{C}_1) \to (X, \upsilon, \mathcal{C}_2)$ be defined as follows : $f(a) = a$, $f(b) = b$, $f(c) = c$.

Here f is somewhat open function but not $\theta - \mathcal{C}$ somewhat open function. Clearly f is open function and hence f is somewhat open function.

In the convex topological space $(X, \upsilon, \mathcal{C}_2)$, $\{a\}_* = X$, $\{b,c\}_* = X$. Thus for $U = \{b,c\} \in \tau$, we have no $V \in \upsilon$ such that $V_* \subseteq f(U_*)$ [since $\{b,c\}_* = \{b,c\}$ in $(X, \tau, \mathcal{C}_1)$]. Hence f is not $\theta - \mathcal{C}$ somewhat open function.

5 Special type of Convex Topological Space

In this article we define a spacial type of convex topological space. Also we introduce new types of CTS on which reverse implications of the diagram-1, as shown in the previous article holds.

Definition 5.1 Let $(X, \tau, \mathcal{C})$ be a convex topological space. The space $(X, \tau, \mathcal{C})$ is called $\tau - \mathcal{C}$ semi compatible if for every $A \in \tau$, A_* is a τ_*-closed set, i.e., if $A \in \tau$, then $(A_*)_* = A_*$.

Example 5.2 Let $X = \{a, b, c\}$, $\tau = \{\Phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{C} = \{\Phi, X, \{a\}\}$. Here $\{a\}_* = X$, $\{b\}_* = \{b, c\}$ and $\{b, c\}_* = \{b, c\}$. Thus $[\{b\}_*]_* = \{b\}_*$. Again $\{a, b\}_* = X$. Thus $(X, \tau, \mathcal{C})$ is $\tau - \mathcal{C}$ semi compatible.

Example 5.3 Let $X = [-1, 1]$. Let τ consists of all those subsets of X, which either do not contain $\{0\}$ or contain $(-1, 1)$. Let $\mathcal{C} = \{\Phi, X\} \cup \{\{x\} : x \in X\}$. Here $(-1, 1)_* = (-1, 1)$, $(-1, 1]_* = (-1, 1]$, $[-1, 1)_* = [-1, 1)$. Let $A \in \tau$ be such that $0 \notin A$. Now $A_* = A \cup \{0\}$ and $[A \cup \{0\}]_* = A \cup \{0\}$. So for every $A \in \tau$, A_* is τ_*-closed. Thus $(X, \tau, \mathcal{C})$ is $\tau - \mathcal{C}$ semi compatible.

Example 5.4 Every normed linear space with usual convexity is $\tau - \mathcal{C}$ semi compatible. In fact, every locally convex space is $\tau - \mathcal{C}$ semi compatible.
Let (X, τ) be a locally convex space. Let us consider the topology τ_* on (X, τ). Again let $A \subseteq X$ be such that $A = \overline{A}$. We will show that $A = A_*$. Let $x \in A_*$ and $x \in U \in \tau$. Since (X, τ) is locally convex space, there exists $V \in \tau$ such that V is convex and $x \in V \subseteq U$. Now $x \in A_* \Rightarrow co(V) \cap A \neq \phi \Rightarrow V \cap A \neq \phi$[since $V = co(V)$] $\Rightarrow U \cap A \neq \phi \Rightarrow x \in \overline{A}$. Consequently $A = \overline{A} = A_*$. Thus in this case we have $\tau = \tau_*$. Hence (X, τ) is $\tau - \mathcal{C}$ semi compatible.

In the following example we will show that not all convex topological space $(X, \tau, \mathcal{C})$ is $\tau - \mathcal{C}$ semi compatible.

Example 5.5 Let $X = \{a, b, c, d, e\}$, $\tau = \{\Phi, X, \{d\}, \{e\}, \{e, d\}\}$, $\mathcal{C} = \{\Phi, X, \{c, d\}\}$. Here $\{e\} \in \tau$. Now $\{e\}_* = \{a, b, c, e\}$ and $(\{e\}_*)_* = X \neq \{e\}_*$. Hence $(X, \tau, \mathcal{C})$ is not $\tau - \mathcal{C}$ semi compatible.

Definition 5.6 A subset G of a CTS $(X, \tau, \mathcal{C})$ is said to be $R - \mathcal{C}$ open if $int(G_*) = G$.

Proposition 5.7 1)In $(X, \tau, \mathcal{C})$ every $R - \mathcal{C}$ open set is open.
2)If $(X, \tau, \mathcal{C})$ is $\tau - \mathcal{C}$ semi compatible, then for every $A \in \tau$, $int(A_*)$ is $R - \mathcal{C}$ open.

Proof:1) Obvious.
2) Let $A \in \tau$ and let $V = int(A_*)$. Since $(X, \tau, \mathcal{C})$ is $\tau - \mathcal{C}$ semi compatible, $(A_*)_* = A$. We will show that $int(V_*) = V$. Now $int(V_*) = int((int(A_*))_*) \subseteq int((A_*)_*) = int(A_*) = V$. Again $V = int(V) \subseteq int(V_*)$. Hence $int(V_*) = V$.

Definition 5.8 A CTS $(X, \tau, \mathcal{C})$ is said to be a

1) $QR-\mathcal{C}$ space if for every nonempty open set G of X, there exists a nonempty open set D such that $D_* \subseteq G$.

2) $SQR-\mathcal{C}$ space if for every nonempty open set G of X, there exists a nonempty $R-\mathcal{C}$ open set A such that $A \subseteq G$.

3) $AQR-\mathcal{C}$ space if for every nonempty $R-\mathcal{C}$ open set $A \subseteq X$, there exists a nonempty open set D such that $D_* \subseteq A$.

Proposition 5.9 A CTS $(X, \tau, \mathcal{C})$ which is $\tau-\mathcal{C}$ semi compatible is a $QR-\mathcal{C}$ space iff it is both $SQR-\mathcal{C}$ space and an $AQR-\mathcal{C}$ space.

Proof : Let $(X, \tau, \mathcal{C})$ is a $QR-\mathcal{C}$ space. Again let G be a nonempty open set of X. Since X is $QR-\mathcal{C}$ space, there exists a nonempty open set D such that $D_* \subseteq G$. Thus $int(D_*) \subseteq G$. Since $int(D_*)$ is $R-\mathcal{C}$ open set [by Proposition 5.7.(ii)] we conclude that X is $SQR-\mathcal{C}$ space.

Again let A be any nonempty $R-\mathcal{C}$ open set. Then A is an open set. Since X is a $QR-\mathcal{C}$ space, there exists a nonempty open set D of X such that $D_* \subseteq A$. Hence X is a $AQR-\mathcal{C}$ space.

To prove the converse, let G be any nonempty open set of X. Since X is $SQR-\mathcal{C}$ space, there exists a nonempty $R-\mathcal{C}$ open set A of X such that $A \subseteq G$. Again since X is a $AQR-\mathcal{C}$ space, there exists a nonempty open set D of X such that $D_* \subseteq A$. Hence $D_* \subseteq A \subseteq G$.Consequently X is a $QR-\mathcal{C}$ space.

Note 5.10 For the proof of the converse part of the above Proposition the requirement, it is evident that $\tau-\mathcal{C}$ semicompatibility of $(X, \tau, \mathcal{C})$ is not necessary.

Theorem 5.11 Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$ be a function. Then the following properties holds :

1) If $(Y, \upsilon, \mathcal{C}_2)$ is $QR-\mathcal{C}$ space, then f is $\theta-\mathcal{C}$ somewhat open iff it is weak somewhat-$\mathcal{C}$ open.

2) If $(X, \tau, \mathcal{C}_1)$ is $\tau-\mathcal{C}$ semi compatible and $AQR-\mathcal{C}$ space, then f is almost somewhat-$\mathcal{C}$ open iff it is weak somewhat-$\mathcal{C}$ open.

3) If $(X, \tau, \mathcal{C}_1)$ is $SQR-\mathcal{C}$ space, then f is somewhat open iff it is almost somewhat-$\mathcal{C}$ open.

Proof : 1) If f is $\theta - \mathcal{C}$ somewhat open then it is clearly [follows from the diagram] weak somewhat-$\mathcal{C}$ open. To prove the converse, let U be any nonempty open set of X. Since f is weak somewhat-$\mathcal{C}$ open function, there exists a nonempty open set V of Y such that $V \subseteq f(U_*)$. Again since Y is $QR - \mathcal{C}$ space, there exists a nonempty open set D of Y such that $D_* \subseteq V$. Thus we have $D_* \subseteq V \subseteq f(U_*)$. Consequently f is a $\theta - \mathcal{C}$ somewhat open.
2) If f is almost somewhat-$\mathcal{C}$ open then clearly it is weak somewhat-$\mathcal{C}$ open. Let A be any nonempty open set of X. Since $A(\neq \phi) \in \tau$ and $A \subseteq A_*$, we have $int(A_*) \neq \phi$. Now $int(A_*)$ is $R - \mathcal{C}$ open set. Since X is $AQR - \mathcal{C}$ space, there exists a nonempty open set D of X such that $D_* \subseteq int(A_*)$. Again since f is weak somewhat-$\mathcal{C}$ open, there exists a nonempty open set W of Y such that $W \subseteq f(D_*)$. Thus we obtain that $W \subseteq f(D_*) \subseteq f(int(A_*))$. Hence f is almost somewhat-$\mathcal{C}$ open function.
3) If f is somewhat open then clearly it is almost somewhat-$\mathcal{C}$ open. To prove the converse, let X be an $SQR - \mathcal{C}$ space. Again let G be any nonempty open set in X. Since X is $SQR - \mathcal{C}$ space, there exists a nonempty $R - \mathcal{C}$ open set A of X such that $A \subseteq G$. Now A is an open set and since f is almost somewhat-$\mathcal{C}$ open, there exists a nonempty open set W of Y such that $W \subseteq f(int(A_*)) = f(A)$[since A is $R - \mathcal{C}$ open, $int(A_*) = A$]. Hence we obtain that $W \subseteq f(A) \subseteq f(G)$. Therefore f is a somewhat open function.

Corollary 5.12 If $(X, \tau, \mathcal{C}_1)$ is $\tau - \mathcal{C}$ semi compatible and $QR - \mathcal{C}$ space, then the following concepts on a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$: somewhat open, almost somewhat open and weak somewhat $\mathcal{C}$ open are equivalent.

Corollary 5.13 If $(X, \tau, \mathcal{C}_1)$ and $(Y, \upsilon, \mathcal{C}_2)$ are $QR - \mathcal{C}$ space and $(X, \tau, \mathcal{C})$ is $\tau - \mathcal{C}$ semi compatible, then the following concepts on a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$: $\theta - \mathcal{C}$ somewhat open, somewhat open, almost somewhat $\mathcal{C}$ open and weak somewhat $\mathcal{C}$ open are equivalent.

6 Properties of $\theta - \mathcal{C}$ somewhat open function

Theorem 6.1 If $f : (X, \tau, \mathcal{C}_1) \to (Y, \upsilon, \mathcal{C}_2)$ and $g : (Y, \upsilon, \mathcal{C}_2) \to (Z, \psi, \mathcal{C}_3)$ are $\theta - \mathcal{C}$ somewhat open functions, then $g \circ f : (X, \tau, \mathcal{C}_1) \to (Z, \psi, \mathcal{C}_3)$ is also $\theta - \mathcal{C}$ somewhat open function.

Proof : Let U be any nonempty open set of X. Then there exists a nonempty $V \in \upsilon$ such that $V_* \subseteq f(U_*)$ because f is $\theta - \mathcal{C}$ somewhat open. Again since g is $\theta - \mathcal{C}$ somewhat open, there exists a nonempty $W \in \psi$ such that $W_* \subseteq g(V_*)$. Hence $W_* \subseteq g(V_*) \subseteq g(f(U_*)) = (g \circ f)(U_*)$. This shows that $g \circ f$ is $\theta - \mathcal{C}$ somewhat open.

Result 6.2 Let $(X, \tau, \mathcal{C})$ be a convex topological space and $A \subseteq X$. Consider the convex topological space $(A, \tau_A, \mathcal{C}_A)$ where τ_A is subspace topology and $\mathcal{C}_A$ is relative convexity on A. For any subset B of A, $(B)_*^{\tau_A} \subseteq B_*$.

Proof : Let $x \in (B)_*^{\tau_A}$ and $x \in U \in \tau$. It is clear that $x \in A$, so $x \in U \cap A$. Now $x \in U \cap A \in \tau_A \Rightarrow co_A(U \cap A) \cap B \neq \phi \Rightarrow co(U \cap A) \cap A \cap B \neq \phi$ [by relative hull formula] $\Rightarrow co(U \cap A) \cap B \neq \phi$ [since $B \subseteq A$] $\Rightarrow co(U) \cap B \neq \phi$ [since $U \cap A \subseteq U$, $co(U \cap A) \subseteq co(U)$] $\Rightarrow x \in B_*$. Hence $(B)_*^{\tau_A} \subseteq B_*$.

Theorem 6.3 If $(X, \tau, \mathcal{C}_1)$ and $(Y, \upsilon, \mathcal{C}_2)$ are convex topological spaces and A is dense in X and $f : (A, \tau_A, \mathcal{C}_{1A}) \rightarrow (Y, \upsilon, \mathcal{C}_2)$ is $\theta - \mathcal{C}$ somewhat open then any extension $F : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \upsilon, \mathcal{C}_2)$is $\theta - \mathcal{C}$ somewhat open function.

Proof : Let U be any nonempty open set of X. Since A is dense in X, $U \cap A \neq \phi$. Now $U \cap A \in \tau_A$ and f is $\theta - \mathcal{C}$ somewhat open. Then there exists a nonempty open set V of Y such that $V_* \subseteq f((U \cap A)_*^{\tau_A})$—1). By the above result we have
$(U \cap A)_*^{\tau_A} \subseteq (U \cap A)_*[U \cap A \subseteq A] \subseteq U_*$. Since F is an extension of f, from 1) we have $V_* \subseteq F(U_*)$. Hence F is $\theta - \mathcal{C}$ somewhat open function.

Theorem 6.4 If $(X, \tau, \mathcal{C}_1)$ and $(Y, \upsilon, \mathcal{C}_2)$ are convex topological spaces and $X = A \cup B$ where $A, B \subseteq X$ and $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \upsilon, \mathcal{C}_2)$ is a function such that the restrictions $f|A$ and $f|B$ are $\theta - \mathcal{C}$ somewhat open, then f is $\theta - \mathcal{C}$ somewhat open function.

Proof : For any nonempty open set U of X, we have $U = X \cap U = (A \cup B) \cap U = (A \cap U) \cup (B \cap U)$. Suppose that $A \cap U \neq \phi$. Since $f|_A$ is $\theta - \mathcal{C}$ somewhat open and $A \cap U (\neq \phi) \in \tau_A$, there exists a nonempty open set V of Y such that $V_* \subseteq f((A \cap U)_*^{\tau_A})$ $\Rightarrow V_* \subseteq f((A \cap U)_*)$[by Result 6.2] $\Rightarrow V_* \subseteq f(U_*)$. This shows that f is $\theta - \mathcal{C}$ somewhat open function.

If $B \cap U \neq \phi$, by using $f|_B$ is $\theta - \mathcal{C}$ somewhat open, we can similarly prove that f is $\theta - \mathcal{C}$ somewhat open.

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