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MULTI-OBJECTIVE OPTIMAL CONTROL PROBLEM

D. K. Bhattacharya & T. E. Aman

Abstract

The paper first considers a general multi-objective optimal control problem and obtains a necessary and sufficient condition for the existence of solution of such a problem. It shows that the solution of such a problem reduces to finding out the solution of a single objective optimal control problem of known type. Next similar investigations are made with fractional multi-objective optimal control problem. Finally, by using the above results, actual solutions are obtained for particular example of multi-objective optimal control problem and that of multi-objective fractional optimal control problem.

1. Introduction:

Single-objective general optimal control problem is well known [1],[5],[15]. Pontryagin's maximum principle gives a necessary condition of optimality [16]. More specific necessary conditions are the Legendre conditions [10]. For general linear quadratic optimal control problem, conditions of optimality are more transparent [11]. Recently such general linear quadratic problem has been studied in a newly developed abstract space by the authors [2]. Special linear quadratic optimal control problems are more interesting from the solution view point [11]. Further, in all such cases, examples are available from physical as well as from biological world.

Again, if we think of constrained optimization problem and linear / nonlinear programming problems, we see that such problems are well studied and moreover vector generalizations of these problems are also well known [4] [9], [12]. These are called vector maximization problem/ non-inferior solution problem /pareto- optimal problem. Further, examples of such problems are available in many branches of science, especially in Economics. Moreover such problems have also been generalized in abstract spaces [3]. Lastly, fractional forms of such vector optimal problems have also been studied [6], [13], [14].

So far as vector generalization of optimal control problems and also of fractional forms of such problems, are concerned, it is noted that examples of such problems may be cited from real world situations. But no attempt is made as yet to study such vector optimal control problems. The paper attempts, for the first time, to formulate such problems, to investigate their solutions, and to find out the actual solutions in suitable examples.

2. Multi- objective control problem and its solution.

2.1. Statement of a multi–objective optimal control problem (MOCP).

Let $\dot{x} = f(x, u)$ be a dynamical system where $x = x(t) \in \Re^n$, $u = u(t) \in U \subset \Re^p$, $U = (a_i < u_i < b_i)$, $t \in [0, t_1]$, $f: \Re^n \times \Re^p \to \Re^n$ is a c^1 – map; the initial condition is $x = x_0$ when $t = 0$. Let the objective function be $J_x(u) = (J_{1x}(u), J_{2x}(u), \ldots J_{mx}(u))$ where

$J_{jx}(u) = \int_0^{t_1} F_j(x, u)\, du$ $(j = 1, 2, \ldots, m)$ (the integral is supposed to exist for each j).

Then the multi-objective optimal control problem is

$$\underset{u}{opt}\, J_x(u) = \underset{u}{opt}\, (J_{1x}(u), J_{2x}(u), \ldots\ldots, J_{mx}(u)),\ \forall\, u \in U.$$

*Corresponding author: Email dkb_math@yahoo.com

2.2. An example of a quadratic multi–objective control problem

Let fish populations x_i (t) (i = 1, 2,…,n) have growth equations given by $\dot{x}_i$ = x_i $f_i(x)$. Let the harvesting efforts be $u_i(t)$. Let the harvest satisfy catch- per unit effort hypothesis [7]. Then the harvested model is given by

$\dot{x}_i(t) = x_i f_i(x) - q_i(t) u_i(t) x_i(t)$, i = 1, 2,…,n,

where q_i denote the catchability coefficients of x_i. Let u = (u_i), $a_i < u_i < b_i$, i = 1, 2, ……, n. Let α = (α_i) be the desired target for harvest x_i. Then the deviation from the target of harvest x is equal to y = q x - α , q = (q_i). Let the performing index be to minimize the sum of two functions, one being $\| q x - \alpha \|^2$ under the weight functions Q(t) = ($Q_{ab}(t)$) and the other one being $\|u\|^2$ under the weight functions R(t) = ($R_{ab}(t)$) ; a , b = 1,2,….,n. So if Q = (Q^k) and R = (R^k), k = 1, 2,….m, (m $\geq$ 2),then there are m performing criteria

$$J_{kx}(u) = \int_0^{t_1} [(qx-\alpha)^T Q^k (qx-\alpha) + u^T R^k u] dt.$$

The problem is to minimize $J_x(u) = (J_{1x}(u), J_{2x}(u), \ldots J_{mx}(u))$, $\forall u \in U$.
This is an example of a (M O C P), where the optimization depends on the choice of u = u* and also on the choice of J_{jx} (i= 1, 2,…, m) and optimization means minimization.

Remark 1: The above problem reduces to a standard optimal control problem if Q and R are taken as identity matrices.

2.3. Geometrical meaning of an optimal control problem and a multi–objective optimal control problem.

Let $\dot{x}$= f(x, u), x$\in \Re^n$, u $\in U \subset \Re^n$, be a system of ordinary differential equations with initial condition x = x_0, and let J = $\int_0^{t_1}$ F (x, u) dt be the objective function. Let the optimal control problem be to optimize J(x, u) over u. Now the integral J is evaluated along each integral curve of $\dot{x}$= f(x, u), passing through x = x_0 corresponding to different choices of u. So J: X $\to$R, where X is the set of all (x, u) defining J(x, u). The solution of the optimal control problem means the optimal value of the integral along a particular integral curve x (t) = x*(t) through x = x_0 which corresponds to the choice of u = u* (called the optimal control). This is why, for the sake of convenience, we write J as J(x,u) or $J_x(u)$ and optimal J as J(x*,u*) or $J_{x^*}(u^*)$.
In a multiple objective control problem, the objective function is vector valued in nature, due to the presence of some other functions like Q and R, as taken in the above example. Naturally if the vector components of the objective function are taken as $J_{1x}(u)$, $J_{2x}(u)$,……$J_{mx}(u)$, then for one such J_{ix} (u) (i=1,2, …,m), all the objective values corresponding to different choices of u evaluated along different integral curves through x = x_0 may be comparable. In that case, it is meaningful to say that $J_{jx^*}(u^*)$ is the optimal value of the control problem, where x* = x*(t) is that integral curve through x = x_0 (called the optimal x*) which corresponds to the optimal u = u*. But the same u* and x* may not optimize all J_{jx} (u) (i=1, 2, …, m). There are three possibilities: (i) (x*,u*) maximizes all J_{jx} (u) (i=1,2, …,m), (ii) (x*,u*) maximizes some J_{jx} (u) (i=1,2, …,m) and minimizes the rest J_{jx} (u) (iii) (x*,u*) maximizes some J_{jx} (u) (i=1,2, …,m), minimizes some J_{jx} (u) and neither maximizes nor minimizes the rest J_{jx} (u). For meaningful discussion, we consider the first two cases only. On similar ground, A. M. Geoffrion [9] and others gave some meaning to vector maximization problems. We consider similar concepts for our vector control maximization problem.

2.4. Efficient and properly efficient multi-objective control problem.

Definition 1. Let a M O C P be stated as in 2.2 where the problem is a maximization problem. Let X = {(x, u)}, where u $\in$U and x is the integral curve of $\dot{x}$= f(x, u) passing through the initial point x =x_0. Then (x*, u*)$\in$X is said to be an efficient solution of MOC P if there exists no (x, u) such that for all i (i= 1,2,....,m), J_{ix} (u) $\leq$ J_{jx^*} (u*) but there exists at least one l (l =1,....,m) such that J_{lx} (u) < J_{lx^*} (u*), $\forall$ (x, u) $\in$X.

Definition 2.

(x*, u*)$\in$ X is said to be a properly efficient solution of M O C P , if it is efficient in the sense that there exists I = { i$\in$ (1,2,...,m) : J_{ix}(u)> J_{ix^*}(u*)}, L={ l$\in$ (1,2,...,m) : J_{lx^*}(u*) > J_{lx}(u)}, $\forall$ (x, u) $\in$X., (x ,u)$\neq$ (x*, u*), $I \cup L$ = (1,2,..,m) and if there exists a scalar M > 0 such that for each i$\in I$, there exists some l $\in$ L, such that

$$\frac{J_{ix}(u)-J_{ix^*}(u^*)}{J_{lx^*}(u^*)-J_{lx}(u)} \leq M, \forall (x,u) \in X .$$

Definition 3.

(x*, u*) $\in$ X is said to be a k-th entry efficient solution of M O C P, if k$\in$(1,2,..,m) such that when J_{kx} (u) > J_{kx^*} (u*), $\forall$(x,u) $\in$X., then there exists at least one l$\in \hat{K}$=(1,2, ... , k-1,k+1, .. .m) for which J_{lx^*}(u*) > J_{lx}(u)}, $\forall$ (x, u) $\in$X.

Definition 4.

(x*, u*) $\in$ X is said to be a properly k-th entry efficient solution of M O C P, if it is a k-th entry efficient solution and further if there exists a scalar M_k > 0 such that

$$\frac{J_{kx}(u)-J_{kx^*}(u^*)}{J_{lx^*}(u^*)-J_{lx}(u)} \leq M_k, \forall (x,u) \in X .$$

We readily have the following propositions:

Proposition 1.

(x*, u*) $\in$X is an efficient solution of M O C P if and only if (x*, u*) is a k-th entry efficient solution for each k$\in$ (1, 2,..., m).

Proposition 2.

(x*, u*) $\in$ X is a properly k-th entry efficient solution of M O C P if and only if (x*, u*) is a properly k-th entry efficient solution for each k $\in$ (1, 2,..., m).

Remark:

To discuss an efficient solution or a properly efficient solution of M O C P, it is sufficient to consider a k-th entry efficient solution or a properly k-th entry efficient solution only.

2.5. Scalar maximum optimal control problem (SMCP) and multiple-objective control problem (MOCP).

In general, we can always find a subset of the set of all K-th efficient solutions of MOCP which are also K-th properly efficient solutions. In this connection, we need the idea of k-th entry SMCP (scalar maximization control problem). Such problems consist of problems of the form λ_{MCP}, $\lambda \in R_+^{m-1}$, where a λ_{MCP} is defined as follows:

Let $l \in \hat{K}$ (=1,2,….,k-1, k+1,…….,m). Then the definition of a λ_{MCP} is:

Maximize $[J_{kx}(u) + \sum_{l\in\hat{K}} \lambda_l J_{lx}(u)]$, $\forall$ (x, u) $\in$X.

We now prove the following characterization theorem of k-th entry properly efficient solution of MOCP

Theorem 1.

Every maximum solution of k-th entry SMCP is an efficient solution of k-th entry MOCP. It is also a properly k-th entry efficient solution of the MOCP. Conversely, every properly k-th entry efficient solution of a MOCP is an optimal solution of k-th entry λ_{MCP}, for some $\lambda \in R_+^{m-1}$.

Proof: Let (x^*, u^*, λ^*) be the point of maximum of λ_{MCP}, for some $\lambda^* \in R_+^{m-1}$, then

$J_{kx}(u) + \sum_{l\in\hat{K}} \lambda^*_l J_{lx}(u) \leq J_{kx^*}(u^*) + \sum_{l\in\hat{K}} \lambda^*_l J_{lx^*}(u^*)$,

i.e., $(J_{kx}(u) - J_{kx^*}(u^*)) + \sum_{l\in\hat{K}} \lambda^*_l J_{lx}(u) \leq \sum_{l\in\hat{K}} \lambda^*_l J_{lx^*}(u^*)$.

If $J_{kx}(u) - J_{kx^*}(u^*) > 0$, then it follows that $\sum_{l\in\hat{K}} \lambda^*_l J_{lx}(u) \leq \sum_{l\in\hat{K}} \lambda^*_l J_{lx^*}(u^*)$.

Hence $\sum_{l\in\hat{K}} \lambda^*_l (J_{lx}(u) - J_{lx^*}(u^*)) \leq 0$. As $\lambda^*_l > 0$, so there exists at least one $l \in \hat{K}$ such that $J_{lx}(u) < J_{lx^*}(u^*)$, whenever $J_{kx}(u) - J_{kx^*}(u^*) > 0$. Hence (x^*, u^*) is a k-th entry efficient solution of MOCP.

Now we show that (x^*, u^*) is also a K-th entry properly efficient solution of MOCP. If not, given any $M_k > 0$, we have, $J_{kx}(u) - J_{kx^*}(u^*) > M_k [J_{lx^*}(u^*) - J_{lx}(u)]$, $\forall\, l \in \hat{K}$ and $\forall$(x, u) $\in$X. As $\lambda^*_l > 0$ are given in λ_{MCP}, so we choose accordingly $M_k = (m-1) \max \lambda^*_l$, $l \in \hat{K}$. Then summing up from 1 to m-1, we have,

$J_{kx}(u) - J_{kx^*}(u^*) > \sum_{l\in\hat{K}} \lambda^*_l [J_{lx^*}(u^*) - J_{lx}(u)]$

i.e., $J_{kx}(u) + \sum_{l\in\hat{K}} \lambda^*_l J_{lx}(u) > J_{kx^*}(u^*) + \sum_{l\in\hat{K}} \lambda^*_l J_{lx^*}(u^*)$, $\forall$ (x, u) $\in$X.

.This is a contradiction as (x*, u*) is a maximal solution of λ_{MCP}. Hence (x*, u*) is properly k-th entry efficient of MOCP, with M_k= (m-1) max λ^*_l, l$\in \hat{K}$.

Conversely, let (x*, u*) be a properly k-th entry efficient solution such that

$J_{kx}(u) - J_{kx^*}(u^*) < M_k [J_{lx^*}(u^*) - J_{lx}(u)]$, for at least one l$\in \hat{K}$ and $\forall$(x, u) $\in$X., where for each such l, $J_{lx^*}(u^*)- J_{lx}(u)>0$,l $\in \hat{K}$. Hence if we choose λ^*= (1,1,…., M_k,1,.1) $\in \Re_+^{m-1}$, where M_k occurs in the k-th place, then $J_{kx}(u)- J_{kx^*}(u^*)< \sum_{l\in\hat{K}} \lambda_l^* [J_{lx^*}(u^*) - J_{lx}(u)]$.

i.e., $J_{kx}(u)+ \sum_{l\in\hat{K}} \lambda_l^* J_{lx}(u) < J_{kx^*}(u^*) + \sum_{l\in\hat{K}} \lambda_l^* J_{lx^*}(u^*)$, $\forall$(x, u) $\in$X. Hence (x*, u*, λ*), is an optimal solution for the member λ_{MCP} of SMCP where λ*= (1, 1, …. . , M_k, 1,.1) . This completes the proof.

2.6. Procedure to evaluate optimal solution of MOCP

If MOCP is assumed to possess a properly efficient solution (x*, u*), then necessarily (x*, u*) is a maximum solution of some SMCP. As each such SMCP can be thought of as a single objective optimal control problem, so necessarily (x*, u*) satisfies Pontryagin's maximum principle. Thus a working rule to find out the optimal solution of a MOCP, when it exists, may be expressed in terms of finding out the optimal solution to a suitable SMCP equivalent to the MOCP, by applying Pontryagin's maximum principle. This is illustrated in working out the following example.

<u>**Example**</u>

Let $\dot{x}_i$ = $x_i f_i (x) - q_i u_i x_i$, $f_i(x) = -K_i x_i (x_i - \alpha_i)$, i = 1, 2, be a system of differential equations, q_i are constants, $u_i = u_i(t)$ are parameters, $a_i \le u_i(t) \le b_i$. Let MOCP be to maximize (J_1, J_2) over u, where $J_i(x,u) = \int_0^t [(qx-\alpha)^T Q^j (qx-\alpha) + u^T R^j u]dt$, $\alpha = (\alpha_1, \alpha_2)^T$,

$$Q^j = \begin{pmatrix} Q_{j1} & 0 \\ 0 & Q_{j2} \end{pmatrix}, R^j = \begin{pmatrix} R_{j1} & 0 \\ 0 & R_{j2} \end{pmatrix}, \text{ i = 1, 2 ; j = 1,2.}$$

Solution:

Let (x^*, u^*) be a maximum solution of MOCP. We consider the 2-th entry efficient solution where $J_{1x}(u)-J_{1x^*}(u^*)>0$, $J_{2x^*}(u^*)-J_{2x}(u)>0$ and $J_{1x}(u)-J_{1x^*}(u^*)< M_2 [J_{2x}^*(u^*)- J_{2x}(u)]$, for some $M_2>0$. Then the corresponding member of SMCP which is to be maximized is

Maximize $J'_x(u) = J_{1x}(u) + M_2 J_{2x}(u)$, $M_2 > 0$, subject to $\dot{x}_i = x_i f_i(x) - q_i u_i x_i$ (i = 1, 2).

For symmetry of expressions, we write $J'_x(u) = M_1 J_{1x}(u) + M_2 J_{2x}(u)$, $M_2 > 0$, $M_1 = 1$.

For this problem, the Hamiltonian takes the form

$H = M_i [q_i x_i - \alpha_i)^T Q^j (q_i x_i - \alpha_i) + u_i^T R^j u_i] + p_i [x_i f_i(x) - q_i u_i x_i] = M_i [Q_{ji} (q_i x_i - \alpha_i)^2 + R_{ji} u_i^2]$

$+ p_i [x_i f_i(x) - q_i u_i x_i]$, i= 1, 2, j = 1,2.

where (p_1, p_2) is the co- state vector which is to be determined suitably.

Now applying Pontryagin's maximum principle, we get

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -M_i[2q_i Q_{ji}(q_i x_i - \alpha_i)] - p_i\left[x_i \frac{\partial f_i(x)}{\partial x_i} + f_i(x) - q_i u_i\right]$$

$$= -2M_i q_i (Q_{1i} + Q_{2i})(q_i x_i - \alpha_i) - p_i\left[x_i \frac{\partial f_i(x)}{\partial x_i} + f_i(x) - q_i u_i\right]$$

For the equilibrium solution, we have

$u_i = \dfrac{f_i(x)}{q_i}$, i =1, 2 (1)

For steady state solution, we use $-K_i x_i (x_i - \alpha_i) = 0$, i = 1, 2 and obtain

$\dot{p}_i = A_i p_i + B_i$, $A_i = -K_i x_i$, $B_i = -2M_i q_i (Q_{1i} + Q_{2i})(q_i x_i - \alpha_i)$ (2)

Solving (2), we have, as a particular solution,

$p_i = -2M_i q_i (Q_{1i} + Q_{2i})(q_i x_i - \alpha_i)$ (3)

Again for maximum H, $\dfrac{\partial H}{\partial u_i} = 0$, for some $u^* \in (a_i, b_i)$. From this, it follows that

$2(R_{11}+R_{21}) u_1^* - p_1 q_1 x_1 = 0$, $2(R_{12}+R_{22}) u_2^* - p_2 q_2 x_2 = 0$ (4)

From (1), u* is given by $u_i^* = f_i(x^*)/q_i$ (5)

Using the values of p_i from (3) and u^*_i from (5) in (4), we have the optimal value of $x^* = (x^*_1, x^*_2, x^*_3)$ given by positive roots of the equations

$$(R_{11}+R_{21})K_1(\alpha_1 - x_1^*) - q_1^{\ 3}(x_1^*)\ M_1(Q_{11}+Q_{21})(q_1x_1^* - \alpha_1) = 0$$
$$(R_{12}+R_{22})K_2(\alpha_2 - x_2^*) - q_2^{\ 3}(x_2^*)\ M_2(Q_{12}+Q_{22})(q_2x_2^* - \alpha_2) = 0 \quad (6)$$

under suitable choice of the parameters. Using this value of x* in (5), we get the optimal u*. Thus (5) and (6) determine the optimal solution (x*, u*) of M O C P.

3. Multi- objective fractional optimal control problem (M O F C P).

3.1. Statement of Multi- objective optimal fractional control problem (M O F C P).

Let $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,u) \\ g(y,u) \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be two systems of ordinary differential equations, where $x = (x_i)$, $y = (y_i)$, i = 1,2, … ,n. Let the objective function be given by $J_{xy}(u)=\{(J_{ixy}(u)\}$ where $J_{ixy}(u) = \dfrac{p_i(x,u)}{q_i(y,u)}$, i = 1,2, … ,m; ($q_{iy}(u) \neq 0$), u being the control parameter, $p_{ix}(u)$ and $q_{iy}(u)$ being given by $p_{ix}(u) = \int_0^{t_1} F_i(x, u)\, dt$, $q_{iy}(u) = \int_0^{t_1} G_i(y,u)\, dt$. Let XY denote the set of all (x, y, u) for which $J_{ixy}(u)$ is defined. Then MOFCP is defined as

Maximize $J_{ixy}(u)$, (i=1,2, .. , m), $\forall$ u $\in$ U.

3.2. Meaning of maximal solution of MOFCP

It is noted that if (x*, y*, u*) maximizes some $J_{ixy}(u) = \dfrac{p_{ix}(u)}{q_{iy}(u)}$, (i = 1, 2, …..m), then it may be assumed that $p_{ix^*}(u^*)$ is the maximum and $q_{iy^*}(u^*)$ is the minimum value of $p_{ix}(u)$ and $q_{iy}(u)$ respectively. But the same (x*, y*, u*) may not maximize all $J_{ixy}(u)$. Hence to make the MOFCP meaningful, we use a modified definition of efficient and properly efficient point. These are parallel to the corresponding definitions which we have considered in case of MOCP. But these are completely new compared to the earlier definitions in the similar cases as considered by E. U. Choo [8] and others.

Definition 5.

$(x^*,y^*,u^*) \in XY$ is called an efficient point of MOFCP, if there exists no $(x, y, u) \in XY$ such that for all i = 1, 2, .., m, $p_{ix}(u) > p_{ix^*}(u^*)$ and $q_{iy}(u) < q_{iy^*}(u^*)$, but there exists at least one l (l =1,2, , .., m) such that $p_{lx}(u) < p_{lx^*}(u^*)$ and $q_{ly}(u) > q_{ly^*}(u^*), \forall(x,y,u) \in XY$.

Definition 6.

$(x^*,y^*,u^*) \in XY$ is called a properly efficient point of MOFCP , if (x^*,y^*,u^*) is efficient in the sense that there exists $I = \{i : p_{ix}(u) > p_{ix^*}(u^*) , q_{iy}(u) < q_{iy^*}(u^*)\}, \forall(x,y,u) \in XY$, and $L = \{ l: p_{lx}(u) < p_{lx^*}(u^*), q_{ly}(u) > q_{ly^*}(u^*)\}, \forall(x,y,u) \in XY$, and further if there exists a constant M>0 such that for each k in I, there exists one $l \in \hat{K} = (1,2,..,k-1.k+1,..,m)$ for which

$$\left[\frac{p_{kx}(u)-p_{kx^*}(u^*)}{q_{ly}(u)-q_{ly^*}(u^*)}\right]\left[\frac{q_{ky^*}(u^*)-q_{ky}(u)}{p_{lx^*}(u^*)-p_{lx}(u)}\right] \leq M .$$

Definition 7.

$(x^*,y^*u^*) \in XY$ is said to be a k-th entry efficient solution of M O C P, if $k \in (1,2,..,m)$ such that when $p_{kx}(u) > p_{kx^*}(u^*)$, $q_{ky}(u) < q_{ky^*}(u^*)\}$, $\forall(x,y,u) \in XY$, then there exists at least one $l \in \hat{K}$ =(1,2,…,k-1,k+1,.., m) for which $\{p_{lx}(u) < p_{lx}(u^*), q_{ly}(u) > q_{ly^*}(u^*)\}, \forall(x, y, u) \in XY$.

Definition 8.

$(x^*, y^*,u^*) \in XY$ is said to be a properly k-th entry efficient solution of M O FC P, if it is a k-th entry efficient solution and further if there exists a scalar $M_k > 0$ such that

$$\left[\frac{p_{kx}(u)-p_{kx^*}(u^*)}{q_{ly}(u)-q_{ly^*}(u^*)}\right]\left[\frac{q_{ky^*}(u^*)-q_{ky}(u)}{p_{lx^*}(u^*)-p_{lx}(u)}\right] \leq M_k, l \in \hat{K} .$$

We readily have the following propositions:

Proposition 3.
$(x^*, y^*,u^*) \in XY$ is an efficient solution of MOFC P if and only if (x^*, y^*, u^*) is a k-th entry efficient solution for each $k \in (1,2,\ldots,m)$.

Proposition 4.
$(x^*,y^*, u^*) \in XY$ is a properly k-th entry efficient solution of MOFC P if and only if (x^*, y^*, u^*) is a properly k-th entry efficient solution for each $k \in (1,2,\ldots,m)$.

Remark:
To discuss an efficient solution or a properly efficient solution of M OFC P, it is sufficient to consider its k-th entry efficient solution or a properly k-th entry efficient solution only.

3.2. An example

Let there be n varieties of fishes, for each of which, a fixed age is considered as its mature stage and a fixed age is considered as its immature stage .Naturally the growth rate of mature ones differ from that of immature ones. Obviously for the mature ones, natural growth rate is less and loss due to intra-specific coefficient is also less; whereas both are higher for immature ones. Let us denote the mature and immature varieties by x = (x_i) and y = (y_i), respectively, i= 1,2,....n . Let their growth equations under harvesting u=u(t) for time interval 0 to t_1 be given by

$\dot{x}$= f(x, u), x (0) = x_0

$\dot{y}$ = g(y, u), y (0) = y_0.

Let the profits of selling i-th varieties of x and y (i= 1, 2,..., n) be given respectively by

$p_{ix}(u) = \int_0^{t_1} F_i(x, u)\,dt$, $q_{iy}(u) = \int_0^{t_1} G_i(y, u)\,dt$

Let the problem be to compare the ratios of $p_{ix}(u)$ and $q_{iy}(u)$ for i= 1,2,....,n ($q_{iy}(u) \neq 0$) and to find the optimal one.
This is an example of a multi-objective fractional optimal control problem MOFCP.

3.3. Scalar maximum fractional optimal control problem (SMFCP) and multiple-objective fractional optimal control problem (MOFCP).

In general, we can always find a subset of the set of all K-th efficient solutions of MOFCP which are also K-th properly efficient solutions. In this connection, we need the idea of k-th entry SMFCP (scalar maximization fractional control problem). Such problems consist of problems of the form $\lambda\mu_{MCP}$, where a $\lambda\mu_{MCP}$ is defined as follows:

Let $\lambda, \mu \in R_+^{m-1}$, k ∈ (1,2,..., m). Let $l \in \hat{K}$(=1,2,....,k-1, k+1,.......,m). Then $\lambda\mu_{MCP}$ is to maximize $[p_{kx}(u) - q_{ky}(u) + \sum_{l\in\hat{K}} (\lambda_l p_{lx}(u) - \mu_l q_{ly}(u))$, ∀ (x, y, u) ∈ XY.

We now prove the following characterization theorem of k-th entry properly efficient solution of MOFCP

Theorem 3.

Every maximum solution of k-th entry SMFCP is a k-th entry efficient solution of MOFCP. It is also a properly k-th entry efficient solution of the MOFCP. Conversely, every properly k-th entry efficient solution of a MOFCP is a k-th entry maximum solution of some $\lambda\mu_{MCP}$, $\lambda,\mu \in R_+^{m-1}$.

Proof: Let (x*, u*, λ*, μ*) be the point of maximum of $\lambda\mu_{MCP}$; then

$[p_{kx}(u) – q_{ky}(u)] + \sum_{l \in \hat{K}} (\lambda^*_l p_{lx}(u) - \mu^*_l q_{ly}(u))$

$\leq [p_{kx^*}(u^*) – q_{ky^*}(u^*)] + \sum_{l \in \hat{K}} (\lambda^*_l p_{lx^*}(u^*) - \mu^*_l q_{ly^*}(u^*))$

i.e., $[p_{kx}(u) - p_{kx^*}(u^*)] + [q_{ky^*}(u^*) - q_{ky}(u)] + \sum_{l \in \hat{K}} (\lambda^*_l p_{lx}(u) - \mu^*_l q_{ly}(u))$

$\leq \sum_{l \in \hat{K}} (\lambda^*_l p_{lx^*}(u^*) - \mu^*_l q_{ly^*}(u^*))$.

Let $p_{kx}(u) - p_{kx^*}(u^*) > 0$ and $q_{ky^*}(u^*) - q_{ky}(u) > 0$, for some $k \in (1,2, .. , m)$ and $\forall$ (x, y, u) $\in$ XY,

then $\sum_{l \in \hat{K}} (\lambda^*_l p_{lx}(u) - \mu^*_l q_{ly}(u)) \leq \sum_{l \in \hat{K}} (\lambda^*_l p_{lx^*}(u^*) - \mu^*_l q_{ly^*}(u^*))$, $\forall$ (x, y, u) $\in$ XY.

Hence $\sum_{l \in \hat{K}} \lambda_{l^*} (p_{lx}(u) - p_{lx^*}(u^*)) + \sum_{l \in \hat{K}} \mu^*_l (q_{ly^*}(u^*) - q_{ly}(u)) \leq 0$.

As $\lambda^*_l, \mu^*_l > 0$, so there exists at least one $l \in \hat{K}$ such that $p_{lx}(u) - p_{lx^*}(u^*) < 0$ and

$q_{ly^*}(u^*) - q_{ly}(u) < 0$, whenever $p_{kx}(u) - p_{kx^*}(u^*) > 0$ and $q_{ky^*}(u^*) - q_{ky}(u) > 0$.

Hence (x*, y*, u*) is a k-th entry efficient solution of MOFCP.

Now we show that (x*, y*, u*) is a K-th entry properly efficient solution. If not, given any $M_k > 0$, we have,

$$\left[\frac{p_{kx}(u) - p_{kx^*}(u^*)}{q_{ly}(u) - q_{ly^*}(u^*)}\right]\left[\frac{q_{ky^*}(u^*) - q_{ky}(u)}{p_{lx^*}(u^*) - p_{lx}(u)}\right] > M_k, \forall\, l \in \hat{K}, \forall\, (x, y, u) \in XY,$$

If we write $M_k = m_k n_k$, then we can take $[p_{kx}(u) - p_{kx^*}(u^*)] > m_k [p_{lx^*}(u) - p_{lx}u)]$ and

$[q_{ky^*}(u^*) - q_{ky}(u)] > n_k [q_{ly}(u) - q_{ly^*}(u^*)]$. As $\lambda_{l^*}, \mu^*_l > 0$ are given in $\lambda\mu_{MCP}$, so we choose

$m_k = (m-1) \max \lambda^*_l$ and $n_k = (m-1) \max \mu^*_l$, $l \in \hat{K}$. Now summing up from 1 to m-1 gives

$[p_{kx}(u) - p_{kx^*}(u^*)] > \sum_{l \in \hat{K}} \lambda^*_l [p_{lx^*}(u) - p_{lx}u)]$ and $[q_{ky^*}(u^*) - q_{ky}(u)] > \sum_{l \in \hat{K}} \mu^*_l [q_{ly}(u) - q_{ly^*}(u^*)]$.

Adding we get

$[p_{kx}(u) - p_{kx^*}(u^*)] + [q_{ky^*}(u^*) - q_{ky}(u)] > \sum_{l \in \hat{K}} \lambda^*_l [p_{lx^*}(u) - p_{lx}(u)] + \sum_{l \in \hat{K}} \mu^*_l [q_{ly}(u) - q_{ly^*}(u^*)]$.

i.e, $[p_{kx}(u) – q_{ky}(u)] + \sum_{l \in \hat{K}} (\lambda^*_l p_{lx}(u) - \mu^*_l q_{ly}(u))$

$> [p_{kx^*}(u^*) – q_{ky^*}(u^*)] + \sum_{l \in \hat{K}} (\lambda^*_l p_{lx^*}(u^*) - \mu^*_l q_{ly^*}(u^*))$, $\forall$ (x, y, u) $\in$ XY, This shows that (x*,y*,u*) is not a solution of $\lambda\mu_{MCP}$, $\lambda,\mu \in R_+^{m-1}$. This is a contradiction. Hence (x*, y*, u*) is a K-th entry properly efficient solution.

Conversely, let (x*, y*, u*) be a properly k-th entry efficient such that

$$\left[\frac{p_{kx}(u)-p_{kx*}(u^*)}{q_{ly}(u)-q_{ly*}(u^*)}\right]\left[\frac{q_{ky*}(u^*)-q_{ky}(u)}{p_{lx*}(u^*)-p_{lx}(u)}\right]\leq M_k, \forall (x,y,u)\in XY, \text{ for at least one } l\in\hat{K}, \text{ where}$$

each of $p_{lx*}(u) - p_{lx}u)$ and $q_{ly}(u)- q_{ly*}(u^*)]$ is positive, $l\in\hat{K}$. Now taking $M_k= m_k n_k$, we get,

$[p_{kx}(u)- p_{kx*}(u^*)] \leq m_k [p_{lx*}(u)- p_{lx}u)]$ and $[q_{ky*} (u^*)- q_{ky}(u)] \leq n_k [q_{ly}(u)- q_{ly*}(u^*)]$.

Hence if we choose $\lambda^*= (1,1,\ldots, m_k, 1,.1)$, $\mu^*= (1,1,, n_k, 1,.1)$, $\in \Re_+^{m-1}$,where m_k, n_k occurs in the k-th place of λ^* and μ^* respectively, then

$[p_{kx}(u)- p_{kx*}(u^*)] < \sum_{l\in\hat{K}} \lambda_l^* [p_{lx*}(u)- p_{lx}u)]$, $[q_{ky*} (u^*)- q_{ky}(u)] < \sum_{l\in\hat{K}} \mu_l^* [q_{ly}(u)- q_{ly*}(u^*)]$.

So finally we get, $[p_{kx}(u)–q_{ky}(u)]+ \sum_{l\in\hat{K}} (\lambda^*_l p_{lx}(u)- \mu^*_l q_{ly}(u))$

$$\leq [p_{kx*}(u^*) –q_{ky*}(u^*)]+ \sum_{l\in\hat{K}} (\lambda^*_l p_{lx*}(u^*) - \mu^*_l q_{ly*}(u^*)$$

Hence (x*, u*, λ^*,μ^*), is a maximum solution of the member $\lambda\mu_{MCP}$ of SMFCP where $\lambda^*= (1,1,\ldots, m_k, 1,.1)$, $\mu^*= (1,1,, n_k, 1,.1)$, This completes the proof of the theorem.

3.4. Procedure to evaluate optimal solution of MOFCP

If MOFCP is assumed to possess a properly efficient solution (x*, y*, u*), then necessarily (x*, y*, u*) is a maximum solution of some SMFCP. As each such SMFCP can be thought of as a single objective optimal control problem, so necessarily (x*, y*, u*) satisfies Pontryagin's maximum principle. Thus a working rule to find out the optimal solution of a MOFCP, when it exists, reduces to finding out the solution to a suitable SMFCP equivalent to the MOFCP by applying Pontryagin's maximum principle This is illustrated in working out the following example.

Example

Let $\dot{x}_i = x_i f_i(x) - r_i u_i x_i = K_i x_i(\alpha_i - x_i) - r_i u_i x_i$,

$$\dot{y}_i = y_i g_i(y) - s_i u_i y_i = K'_i y_i(\beta_i - y_i) - s_i u_i y_i, \text{ i= 1,2,}$$

be two systems of differential equations, α_i, β_i , r_i, s_i are constants,

$p_j(x,u) = \int_0^{t_1} [x^T Q^j x + u^T R^j u]dt$, $q_j(y,u) = \int_0^{t_1} [y^T Q'^j y + u^T R'^j u]dt$, i= 1,2, j = 1, 2;

$u_i = u_i(t)$, are parameters, $a_i \le u_i(t) \le b_i$, each of Q^1, Q^2, R^1 and R^2 is a 2x2 matrix given by

$Q^j = \begin{pmatrix} Q_{j1} & 0 \\ 0 & Q_{j2} \end{pmatrix}, R^j = \begin{pmatrix} R_{j1} & 0 \\ 0 & R_{j2} \end{pmatrix}, Q'^j = \begin{pmatrix} Q'_{j1} & 0 \\ 0 & Q'_{j2} \end{pmatrix}, R'^j = \begin{pmatrix} R'_{j1} & 0 \\ 0 & R'_{j2} \end{pmatrix}$, Let XY be the set where $p_{ix}(u)$ and $q_{iy}(u)$ are both defined; Let the MOFCP be defined as follows:

Maximize $J_{ixy}(u) = \dfrac{p_{ix}(u)}{q_{iy}(u)}$, i = 1,2, $\forall u \in U$.

Solution:

Let (x*, y*, u*) be a maximum solution of MOFCP. We consider the 2-th entry proper efficient solution, such that there exists $m_k > 0$ and $n_k > 0$ satisfying $[p_{1x}(u) - p_{1x^*}(u^*)] < m_2 [p_{2x^*}(u) - p_{2x}u)]$ and $[q_{1y^*}(u^*) - q_{1y}(u)] < n_2 [q_{2y}(u) - q_{2y^*}(u^*)]$, $\forall (x, y, u) \in XY$, where $\{p_{1x}(u) > p_{1x^*}(u^*), q_{1y}(u) < q_{1y^*}(u^*)\}$, $\{p_{2x}(u) < p_{2x}(u^*), q_{2y}(u) > q_{2y^*}(u^*)\}$.

Then the corresponding member of SMFCP which is to be maximized is the following:

Maximize $J'_{xy}(u) = \sum_{i=1}^{i=2} (m_i p_{ix}(u) - n_i q_{iy}(u))$; $m_2 > 0$, $n_2 > 0$, $m_1 = 1$, $n_1 = 1$.

subject to $\dot{x}_i = K_i x_i(\alpha_i - x_i) - r_i u_i x_i$ $\dot{y}_i = K'_i y_i(\beta_i - y_i) - s_i u_i y_i$

For this problem, Hamiltonian is taken as

$H = m_i(x^T Q^j x + u^T R^j u) - n_i(y^T Q'^j y + u^T R'^j u) + \lambda_i [x_i f_i(x) - r_i u_i x_i + \lambda'_i y_i g_i(y) - s_i u_i y_i$

$= m_i(x^T Q^j x + u^T R^j u) - n_i(y^T Q'^j y + u^T R'^j u)$

$+ \lambda_i[K_i x_i(\alpha_i - x_i) - r_i u_i x_i] + \lambda'_i K'_i y_i(\beta_i - y_i) - s_i u_i y_i$ i = 1, 2, j = 1, 2.

where (λ_i, λ'_i) is the co- state vector to be determined suitably.

Now applying Pontryagin's maximum principle and simplifying, we get

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -m_i[2Q_{ji}x_i] - \lambda_i\left[x_i \frac{\partial f_i(x)}{\partial x_i} + f_i(x) - r_i u_i\right] = -m_i[2Q_{ji}x_i] - K_i x_i \lambda_i$$

$= -2m_i(Q_{1i}+Q_{2i})x_i$ - $K_i x_i \lambda_i = C_i \lambda_i + D_i$; $C_i = -K_i x_i$, $D_i = -2m_i(Q_{1i}+Q_{2i})x_i$

$$\dot{\lambda}'_i = -\frac{\partial H}{\partial y_i} = +n_i[2Q'_{ji}y_i\ ] - \lambda'_i\left[y_i\frac{\partial g_i(y)}{\partial y_i} + g_i(y) - s_i u_i\right] = n_i[2Q'_{ji}y_i\ ] - K'_i y_i \lambda'_i$$

$= 2n_i(Q'_{1i}+Q'_{2i})y_i - K'_i y_i \lambda'_i = C'_i\lambda_i + D'_i$; $C'_i = -K'_i y_i$, $D'_i = 2n_i(Q'_{1i}+Q'_{2i})y_i$.

As for steady state solution, we get $K_i x_i(\alpha_i - x_i) - r_i u_i x_i = 0$, I = 1,2 , so we obtain

$\dot{\lambda}_i = C_i \lambda_i + D_i$; $C_i = -K_i x_i$, $D_i = -2m_i(Q_{1i}+Q_{2i})x_i$ (7)

Solving (7), we have, as a particular solution,

$\lambda_i = -2m_i(Q_{1i}+Q_{2i})x_i$ (8)

Again for maximum H, $\frac{\partial H}{\partial u_i}$=0, for some u* $\in (a_i, b_i)$. From this, it follows that

$2(R_{11}+R_{21})u_1^* - \lambda_1 r_1 x_1 = 0$, $2(R_{12}+R_{22})u_2^* - \lambda_2 r_2 x_2 = 0$ (9)

Considering equilibrium solution as the optimal solution, we have u* given by

$u_i^* = f_i(x^*)/ri = gi(y)/s_i$ (10)

Using the values of λ_i from (8) and u^*_i from (10) in (9), we have optimal $x^* = (x^*_1, x^*_2, x^*_3)$ given by the positive roots of the equations (under some restrictions on the parameters)

$$(R_{11}+R_{21})K_1(\alpha_1 - x_1^*) - r_1^3(x_1^*)\ m_1(Q_{11}+Q_{21})x_1^* = 0$$
$$(R_{12}+R_{22})K_2(\alpha_2 - x_2^*) - r_2^3(x_2^*)\ m_2(Q_{12}+Q_{22})x_2^* = 0 \quad (11)$$

Proceeding similarly we find the optimal value of $y^* = (y^*_1, y^*_2, y^*_3)$ given by the positive roots of the equations (under certain restrictions on the parameters)

$$(R_{11}+R_{21})K'_1(\beta_1 - y_1^*) + s_1^3(y_1^*)\ n_1(Q'_{11}+Q'_{21})y_1^* = 0$$
$$(R_{12}+R_{22})K'_2(\beta_2 - y_2^*) + s_2^3(y_2^*)\ n_2(Q'_{12}+Q'_{22})y_2^* = 0 \quad (12)$$

Now using the value of either x* given by (11) or y* given by (12), we can find the value of u* from (10). Thus the given 2 th entry properly efficient MOFCP is solved out.

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