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An elementary explicit example of unbounded limit behaviour on the plane

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Abstract. An academic new example of two-dimensional planar dynamical system is constructed to describe a very well-known fact. Namely, that unbounded semiorbits could generate nonconnected ω-limit sets (see [Hl], page 48).

1. Introduction. The concept of ω-limit set was introduced by D. Birkhoff (cf. [Bi]). For a differential equation in $\mathbf{R}^n$

$$x' = f(x) \tag{1}$$

the ω-limit set of a certain solution or semiorbit "summarizes", roughly speaking, the asymptotic behaviour of such a solution. If $x = x(t)$ is a solution to (1) defined in $t \geq t_o$, a $z \in \mathbf{R}^n$ is said to be an ω-limit point of $x(t)$ (see [Bi], [Hl]) if there exists $\{t_n\}$, $t_n \to \infty$, such that $x(t_n) \to z$. For a fixed solution $x = x(t)$, or better, for the semiorbit $\gamma = \{x(t)/t \geq t_o\}$ attached to $x = x(t)$, $\Lambda^+(\gamma)$ usually designates the set of ω-limit points of γ. As a possible reliable picture of a physical phenomenom it is clear that a major emphasis must be put in the study of $\Lambda^+(\gamma)$ when γ is a *bounded* semiorbit to (1). In bounded regions of $\mathbf{R}^2$ with finitely many critical singularities of (1), the structure of $\Lambda^+(\gamma)$ for semiorbits γ liying in such regions, is given by the celebrated Poincaré-Bendixon Theorem (see [Ha],[Hl]). In $\mathbf{R}^n$ the more general information about $\Lambda^+(\gamma)$, for γ bounded, is that contained in the next result (see for instance [Hl] page 47)

THEOREM.

If $\gamma = \{x(t)/t \geq t_o\} \subset \mathbf{R}^n$ is a bounded semiorbit to (1) then $\Lambda^+(\gamma)$ is a nonempty, invariant, compact and connected set.

Even in $\mathbf{R}^3$ it is sometimes hardly possible to add a bit more to the general asserts given above about $\Lambda^+(\gamma)$ (see for instance the Lorenz's system in [GH]).

It is also well-known that the connectedness of $\Lambda^+(\gamma)$ is a consequence of the boundedness of γ. Here we will focuss our attention in this precise fact. When $\Lambda^+(\gamma)$ contains two points z_1, z_2, the semiorbit γ will meet infinitely many times every pair of arbitrarily small neighbourhoods U_1, U_2 of z_1 and z_2 (respectively). Boundedness of γ will imply that z_1 and z_2 will be "connected" into $\Lambda^+(\gamma)$. The objective of this note is giving an *explicit* example in $\mathbf{R}^2$ that such connectedness is lost when γ is unbounded. Obviously, this fact is well-known since long (see for instance [Hl] page 48). Moreover, after thinking on it for a while, it is not difficult to arrive to the conclusion that a picture of such an orbit γ shuold be more or less as shown in figure 1.

What is presented in this work is a class of equations that make precise in an explicit and analytic way this kind of behaviour.

2.The results.

Let $\varphi_0 = \varphi_o(y) \in C^1([0,+\infty))$ such that $\varphi_o(0) = 0$, $\varphi_o(y) > 0$ in $y > 0$, and also that $\lim_{y\to+\infty} \varphi_o(y) = 0$. Suppose, without loss of generality, that $\max_{y\geq 0} |\varphi_o(y)| < 1$. Designate by $\varphi = \varphi(y)$ the odd extension of φ_o and call $\alpha = \max_{y\geq 0} |\varphi_o(y)|$. A simple example of such a function is

$$\varphi(y) = \frac{y}{1+y^2}.$$

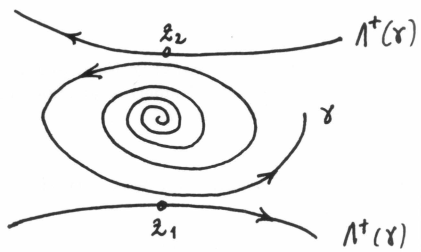

figure 1

Consider now the following class of equations

$$\begin{cases} x' = y(1-x^2) \\ y' = \varphi(y) - x. \end{cases} \tag{2}$$

It will be shown that the ω-limit set of every semiorbit γ starting at $(x_o, y_o) \neq (0,0)$, $x_o^2 < 1$, exactly consits of the lines $\{x = 1\} \cup \{x = -1\}$. Let us first introduce the following regions,

$$I = \{(x,y)/y > 0, \varphi(y) > x\},$$
$$II = \{(x,y)/y > 0, \varphi(y) < x\},$$
$$III = \{(x,y)/y < 0, \varphi(y) < x\},$$
$$IV = \{(x,y)/y < 0, \varphi(y) > x\}.$$

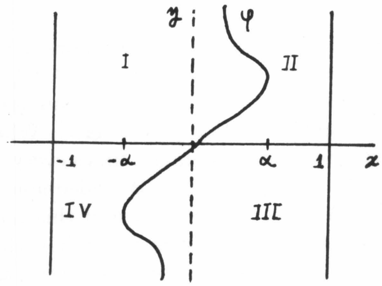

figure 2

The first fact to be proven below is the rotation around $(0,0)$ of the solutions to (2) lying in the strip $(-1,1)\times \mathbf{R}$.

LEMMA 2.1.

Let $(x(t),y(t))$ be a noncontinuable solution to (2) with initial position $(x_o,y_o)\in (-1,1)\times \mathbf{R}$ and maximal existence interval (α,ω). Then $\omega=+\infty$ and $(x(t),y(t))$ pass through the regions I, II, III and IV –following this ordered sequence– arriving again in finite time to I and repeating this sequence infinitely many times.

Remarks.

a) In Lemma 2.1 the initial data (x_o,y_o) could be taken in either of the regions II, III or IV. Then, the orbit starting at (x_o,y_o) will cross trough the remaining regions in a cyclic and ordered way.

b)For $v=(v_1,v_2)$, $v_1\neq 0$, $v_1^2+v_2^2=1$ let us define

$$\Gamma_v=\{(sv_1,sv_2)/s>0\}\cap A$$

where A designates any of the regions introduced above. Γ_v can be ordered in the natural way regarding s. Lemma 2.1 allows us defining the *Poincaré* map π (see [Ha], [Hl]) over Γ_v. In fact, call $(x(t,t_o,x_o,y_o),y(t,t_o,x_o,y_o))$ the (unique) solution to (2) satisfying $(x(0),y(0))=(x_o,y_o)$. If $(x_o,y_o)\in\Gamma_v$ then there exists $\tau(x_o,y_o)=min\{t>0/(x(t,t_o,x_o,y_o),y(t,t_o,x_o,y_o))\in\Gamma_v\}$. From Lemma 2.1 it follows that $\tau(x_o,y_o)$ is defined and positive whatever the choice of $(x_o,y_o)\in\Gamma_v$ be. For $(x_o,y_o)\in\Gamma_v$ define s by the equality $(x_o,y_o)=sv\in\Gamma_v$, write $\tau(s)=\tau(x_o,y_o)$ and define $\sigma=\sigma(s)$ by means of $\sigma v=(x(\tau(s),x_o,y_o),y(\tau(s),x_o,y_o))$. It is well-known that

$$\begin{aligned}\Pi:\quad &\Gamma_v\rightarrow\Gamma_v\\ &sv\rightarrow\sigma(s)v\end{aligned}$$

(the *Poincaré's* mapping) is an increasing C^1 mapping whose (possible) fixed points give rise periodic solutions to (2). In this case it also possible to show that Π is also a C^1 difeomorfism. Notice that s belongs to the interval $(0,\frac{1}{|v_1|})$ provided $\frac{v_2}{v_1}<0$, meanwhile Γ_v could consist on several connected pieces if $\frac{v_2}{v_1}>0$. Designate by $(a(v),\frac{1}{|v_1|})$ that one the most separated from $(0,0)$. For simplicity let us also set $a(v)=0$ when $\frac{v_2}{v_1}$ is negative. We can state the following result,

LEMMA 2.2.

For each $v\in\mathbf{R}^2$, $|v|=1$, $v_1\neq 0$, define the interval $I_v=(a(v),\frac{1}{|v_1|})$ and the C^1 difeomorfism $\begin{aligned}\sigma:\quad &I_v\rightarrow I_v\\ &s\rightarrow\sigma(s)\end{aligned}$. Then, for each $s\in I_v$, $s<\sigma(s)$. Moreover,

a) Equation (2) does not exhibit limit cycles into $(-1,1)\times\mathbf{R}$.

b) For each $s\in I_v$, $\lim\sigma^n(s)=\frac{1}{|v_1|}$

The next result is the objective of this note and is a straightforward consequence of Lemma 2.2.

THEOREM 2.3.

For each (x_o, y_o) *lying in the strip* $(-1,1)\times\mathbf{R}$ *the unique semiorbit* γ *passing through* (x_o, y_o),

$$\gamma = \{(x(t,x_o,y_o), y(t,x_o,y_o))/t \geq 0\}$$

goes off the origin $(0,0)$ *turning around this point infinitely many times and satisfying*

$$\Lambda^+(\gamma) = \{(x,y)/x = 1 or \quad x = -1\}.$$

3. The proofs.

3.1. The proof of Lemma 2.1.

First notice that the strip $(-1,1)\times\mathbf{R}$ is invariant. Indeed, $x=1$ and $x=-1$ are orbits of (2). Assume that $(x_o,y_o)\in I$ and let $(x(t),y(t))$ the solution starting at that point, with maximal existence interval (α,ω), then $(x(t),y(t)$ leaves I at finite time. Otherwise, $(x(t),y(t))\in I$ and $x'(t)>0$, $y'(t)>0$ for each $t\in(\alpha,\omega)$. However, since $x(t)<\alpha$ for each t then $\lim_{t\to\omega} y(t)=+\infty$. Indeed, $y(t)$ can not be bounded. Otherwise, $\omega=+\infty$, which implies $\lim_{t\to+\infty} x'(t)=\lim_{t\to+\infty} y'(t)=0$ what contradicts $x(t)\geq x_o$, $y(t)\geq y_o$ for each t. Therefore, $\lim_{t\to\omega} y(t)=+\infty$. But $\lim_{t\to\omega} y(t)=+\infty$ entails $\omega=+\infty$ since,

$$\begin{aligned} y' &= \varphi(y) - x(t) \\ &\leq \varphi(y) - x_o \\ &\leq \varphi(y) - 1. \end{aligned}$$

and the solutions to $z'=\varphi(z)-1$ do not blow up since $\int_{y_o}^{+\infty}\frac{dy}{\varphi(y)-1}=+\infty$. Therefore we must conclude that $\lim_{t\to+\infty} y(t)=+\infty$. However $\omega=+\infty$ and the fact $x(t)<\alpha$ for $t>t_o$ imply $\lim_{t\to+\infty} x'(t)=0$. On the other hand

$$x'(t) = y(t)(1-x^2) \geq y(t)(1-\alpha^2)$$

for $t>t_o$; and $\lim_{t\to+\infty} y(t)(1-\alpha^2)=+\infty$. Thus $(x(t),y(t))$ must leave I at finite time. In other words, there exits $t_1\in[0,\omega)$ such that $x(t_1)=\varphi(y(t_1))$ which implies $(x(t),y(t))\in II$ for $t=t_1+\epsilon$, and certain positive small enough ϵ. In fact, $x'(t_1)=y(t_1)(1-x^2(t_1))=y(t_1)(1-\varphi(y(t_1))^2)$ is positive and the function $h(t)=x(t)-\varphi(y(t))$ vanishes at $t=t_1$ with $h'(t_1)=x'(t_1)-\frac{d\varphi}{dy}(y(t_1))y'(t_1)=x'(t_1)>0$. Thus, the existence of ϵ is proven.

Next, designate by $(x_1,y_1)=(x(t_1+\epsilon),y(t_1+\epsilon))$. We are going to show the existence of $t_2>t_1+\epsilon$ such that $0<x(t_2)<1$, $y(t_2)=y_2=0$ and so that $(x(t),y(t))\in II$ for each $t_1\leq t<t_2$. To see this, set $K=\{y\leq y_1\}\cap II$. Then, if necessary taking a smaller ϵ, $(x(t),y(t))\in K$ for $t_1\leq t\leq t_1+\epsilon$. We claim that a $t_2>t_1$ exist so that $(x(t_2),y(t_2))\in\partial K$. Otherwise $\omega=+\infty$ and we wolud have $\lim_{t\to+\infty}(x(t),y(t))=(x_2,y_2)$ with $x_2>x_1$. However, monotonicity of both $x(t)$ and $y(t)$ would imply $\lim_{t\to+\infty}(x'(t),y'(t))=(0,0)$, what is not possible. On the other hand, if $t_2=min\{t/t>t_1,(x(t),y(t))\in\partial K\}$ then $(x(t_2),y(t_2))=(x_2,0)$ with $0<x_2<1$. In fact, observe that a part of ∂K consist of $y=y_1$, other one consist of a piece of the graph $x=\varphi(y)$ meanwhile a third part consist of a piece of the orbit $x = 1$. Because of the direction field of (2) the only piece of ∂K where the exit points of $(x(t), y(t))$ can be located is just $y = 0$, as claimed (see figure 3).

Finally, since $y'(t_2) = -x_2 < 0$ there exist $t_2 < t_3 < \omega$ such that $(x(t_3), y(t_3)) \in III$. Thus, it can be asserted that every solution to (2) starting in an arbitrary $(x_o, y_o) \in I$ reachs the region III in a finite period of time t, after passing through the regions I, II. Therefore, it is straightforward to show that every semiorbit γ to (2) starting in $(x_3, y_3) \in III$ also reachs the region I –after passing through III and IV– in a finite period of time. In fact, $\varphi = \varphi(y)$ was chosen in order to (2) were symmetric with respect to $(0,0)$. Specifically, the orbit $\overline{\gamma}$ starting at $(x_o, y_o) = (-x_3, -y_3) \in I$ is the symmetric of γ with respect to $(0,0)$. This is due to the following fact "$(x(t), y(t))$, $t \in (\alpha, \omega)$, is a noncontinuable solution to (2) if and only if $(x_1(t), y_1(t)) = (-x(t), -y(t))$, $t \in (\alpha, \omega)$ is a noncontinuable solution to (2)". So, by the uniqueness of solutions,

$$x(t, -x_o, -y_o) = -x(t, x_o, y_o)$$
$$y(t, -x_o, -y_o) = -y(t, x_o, y_o), \quad t \in (\alpha, \omega).$$

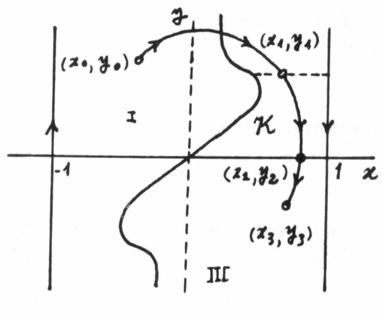

figure 3

Thus, every semiorbit γ starting in $(x_o, y_o) \in I$ finally arrive again in I after crossing I, II, III, IV in finite time, as was claimed.

3.2. The proof of Lemma 2.2.

Firstly, it should be remarked that, in the cases where $\frac{v_2}{v_1} < 0$ the orbits to (2) reach in finite time the connected piece of Γ_v most separated from $(0,0)$. To see this, it suffices with employing the argument to be developed below, and concerning the Lyapunov function $V(x, y) = y^2 - log(1 - x^2)$.

Now observe that for each $v \in \mathbf{R}^2, v_1 \neq 0, |v| = 1$, there exists $s_o \in (a(v), \frac{1}{|v_1|})$ such that $s_o < \sigma(s_o)$. To see this let us observe that by replacing the solution to (2) $(x(t), y(t))$ instead of (x, y) in the (Lyapunov) function

$$V(x, y) = y^2 - log(1 - x^2) \tag{3}$$

we get, after derivation with regard t,

$$\frac{d}{dt}(V(x(t), y(t)) = 2y\varphi(y).$$

On the other hand, observe that $V \geq 0$ and is convex in the strip $(-1,1) \times \mathbf{R}$, $V = 0$ if and only if $y = 0$. In addition, the restriction of V to every segment Γ_v is an increasing function of s, provided $v_2 \neq 0$. Therefore, the existence of a $s_o \in (a(v), \frac{1}{|v_1|})$ such that $s_o < \sigma(s_o)$ follows, in the case $v_2 \neq 0$. By continuity, and using again the behaviour of V the same holds true for the segments Γ_v, $v_2 = 0$. Own existence of V also entails that $\sigma(s) \neq s$ for each $s \in (a(v), \frac{1}{|v_1|})$ and for each $v \in \mathbf{R}^2$, $|v_1| \neq 0$, $|v| = 1$. This fact avoids the possible existence of infinitely many closed orbits to (2), γ_n, whose intersection points with the open segment $(0,1) \times \{0\}$, say $(x_n, 0)$, could converge towards the point $(x,y) = (1,0)$. We should observe that (2) is a perturbation of the equation (4) below (see remark a)), which exhibits a continuum of periodic orbits "filling" the strip $(-1,1) \times \mathbf{R}$. Therefore, we need to rule out the existence of such a family of closed orbits converging to the "sides" $x = \pm 1$ of the strip. On the other hand, since the existence of a $s_1 \in (a(v), \frac{1}{|v_1|})$ such that $s_1 < \sigma(s_1)$ would entail –together with the own existence of s_o– the existence of an "intermediate" s_2 so that $\sigma(s_2) = s_2$, such kind of s_1 can not exist. Thus, $s < \sigma(s)$ for each s and so, the sequence $\{\sigma^n(s)\}$ is always increasing whatever the values of $s \in (a(v), \frac{1}{|v_1|})$ and v be. As the limit $\hat{s} = \lim \sigma^n(s)$ exists it must necessarily be $\hat{s} = \frac{1}{|v_1|}$, otherwise $\sigma(\hat{s}) = \hat{s}$, what is not possible.

Remarks

a) Equation (2) is a perturbation of the equation

$$\begin{cases} x' = y(1-x^2) \\ y' = -x. \end{cases} \tag{4}$$

Equation (4) exhibits a continuum of closed orbits filling the strip $(-1,1) \times \mathbf{R}$ and surrounding $(0,0)$. This can be easyly checked by using the first integral $V(x,y) = y^2 - log(1-x^2)$.

b) If we call $U(x,y)$ the right hand side of (3) –what is usually coined as the derivative of V with regard to equation (2)– it is checked that the single point $\{(0,0)\}$ is the invariant maximal set contained into $\{U(x,y) = 0\}$. Therefore, by reserving the time t in (2) and using the La Salle's invarianze principle (see [Hl], Theorem 1.3 page 316) we can complete the picture of the phase portrait of (2) by asserting that

$$\lim_{t \to -\infty} (x(t, x_o, y_o), y(t, x_o, y_o)) = (0,0),$$

for each $(x_o, y_o) \in (-1,1) \times \mathbf{R}$.

c) The assertion concerning $\Lambda^+(\gamma)$ in Theorem 2.3 is an inmediate consequence of the fact $\lim \sigma^n(s) = \frac{1}{|v_1|}$ on every segment Γ_v.

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TAMAÑO MUESTRAL MÍNIMO Y CONTRASTE DE DOS PROPORCIONES BINOMIALES.

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RESUMEN

En este trabajo diseñamos un algoritmo para calcular el mínimo tamaño muestral para el test de dos parámetros binomiales.

Se contrasta $H_0 \equiv p_1 = p_2 = p_0$ frente $H_a \equiv p_1 = p_0 - \Delta$ y $p_2 = p_0 + \Delta$, $0 < \Delta < p_0 \le 1/2$ con nivel de significación (Error del primer tipo) menor que α y función de potencia mayor que 1-β (Error del segundo tipo menor que β). Se proporcionan unas tablas que definen la función de decisión entre las dos hipótesis.

ABSTRACT

In this paper, we design an algorithm to calculate the minimum sample size for the two parameters binomial test.

We test $H_0 \equiv p_1 = p_2 = p_0$ against $H_a \equiv p_1 = p_0 - \Delta$ and $p_2 = p_0 + \Delta$, $0 < \Delta < p_0 \le 1/2$ with level of significance (Type I error) below α and power function above 1-β (Type II error below β). We supply tables that define the decision function between the two hyphotesis.