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$\theta - \mathcal{C}$ - SOMEWHAT CONTINUOUS FUNCTIONS

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Abstract :The aim of this paper is to introduce a new type of function called $\theta - \mathcal{C}$ somewhat continuous function in a convex topological space is introduced earlier. Some characterizations and various basic properties of this type of function are obtained. Also, its relationship with other types of function is investigated. In this paper we have discussed a comparison between a $\theta - \mathcal{C}$ somewhat continuous function and somewhat continuous function.

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1 Introduction

The development of 'abstract convexity' has emanated from different sources in different ways; the first type of development basically banked on generalization of particular problems such as separation of convex sets [3], extremality [4]; [2], or continuous selection [10]. The second type of development lay before the reader such axiomatizations, which, in every case of design, express of particular point of view of convexity. With the view point of generalized topology which enters into convexity via the closure or hull operator, Schmidt[1953] and Hammer[1955], [1963], [1963b] introduced some axioms to explain abstract convexity. The arising of convexity from algebraic operations and the related property of domainfitness received attentions in Birchoff and Frink[1948] Schmidt[1953] and Hammer[1963].

Throughout this paper the axiomatizations as proposed by M. L. J. Van De Vel in his papers in the seventies and finally incorporated in Theory of Convex Structure [12] will be followed.

In [13] the author has discussed 'Topology and Convexity on the same set' and introduced the compatibility of the topology with a convexity on the same underlying set. At the very early stage of this paper we have set aside the concept of compatibility and started just with a triplet(X,$\tau, \mathcal{C}$)and have called it convex topological space only to bring back'compatibility'in another way subsequently. With his compatibility, however, VanDevel has called the triplet (X,$\tau, \mathcal{C}$) a topological convex structure.

It is however seen that in many cases where compatibility is expected our definition serves the purpose.

In this paper, Art 2 deals with some early definitions and in Art 3, we have discussed $\mathcal{D}-\mathcal{C}$ space. Art.4 deals with $\theta-\mathcal{C}$somewhat continuous function and its basic properties. In the last article a new type of convex topological space is introduced which is called strongly $\mathcal{C}$-regular space.

2 Prerequisites

Definition 2.1 [13] Let X be a nonempty set. A family $\mathcal{C}$ of subsets of the set X is called a convexity on X if

1. $\Phi, X \in \mathcal{C}$

2. $\mathcal{C}$ is stable for intersection, i. e. if $\mathcal{D} \subseteq \mathcal{C}$ is nonempty then $\cap\mathcal{D} \in \mathcal{C}$.

3. $\mathcal{C}$ is stable for nested unions, i. e. if $\mathcal{D} \subseteq \mathcal{C}$ is nonempty and totally ordered by set inclusion then $\cup\mathcal{D} \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convex structure. The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets.

Definition 2.2 [13] Let $\mathcal{C}$ be a convexity on a set X. Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by
$co(A) = \cap\{C : A \subseteq C \in \mathcal{C}\}$.

Note 2.3 [13] Let $(X, \mathcal{C})$ be a convex structure and let Y be a subset of X. The family of sets $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a convexity on Y; it is called the relative convexity of Y.

Note 2.4 [13] The hull operator co_Y of a subspace $(Y, \mathcal{C}_Y)$ satisfies the following : $\forall A \subseteq Y : co_Y(A) = co(A) \cap Y$.

Definition 2.5 Let (X, τ) be a topological space. Let $\mathcal{C}$ be a convexity on X. Then the triplet $(X, \tau, \mathcal{C})$ is called a convex topological space (CTS, in short).

Theorem 2.6 [5] Let $(X, \tau, \mathcal{C})$ be a convex topological space. Let A be a subset of X. Consider the set A_*, where A_* is defined as follows : $A_* = \{x \in X : co(U) \cap A \neq \phi, x \in U \in \tau\}$. Then the collection $\tau_* = \{A^c : A \subseteq X, A = A_*\}$ is a topology on X such that $\tau_* \subseteq \tau$.

A is said to be τ_*- closed if $A = A_*$.

Definition 2.7 [5] Let $(X, \tau, \mathcal{C})$ be a convex topological space. The space $(X, \tau, \mathcal{C})$ is called τ-$\mathcal{C}$ semi compatible if for every $A \in \tau$, A_* is a τ_*- closed set, i.e., if $A \in \tau$, then $(A_*)_* = A_*$.

Definition 2.8 [5] Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is said to be

1. $\theta - \mathcal{C}$-open if for each $x \in X$ and each nbd. U of x, there exists a nbd. V of $f(x)$ in Y such that $V_* \subseteq f(U_*)$ and

2. $\theta - \mathcal{C}$ somewhat open if $U \in \tau$ and $U \neq \phi$, then there exists a $V \in \sigma$ such that $V \neq \phi$ and $V_* \subseteq f(U_*)$.

3 $\mathcal{D} - \mathcal{C}$ -space

Definition 3.1 A convex topological space $(X, \tau, \mathcal{C})$ is said to be a $\mathcal{D} - \mathcal{C}$ -space if every nonempty open subset of X is τ_* dense in X i.e. if $A(\neq \phi) \in \tau$ then $A_* = X$.

Note 3.2 From the above definition it follows that if a subset A is τ dense in X i.e. $\overline{A} = X$ then it is automatically τ_* dense in X.

Theorem 3.3 If a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is $\theta - \mathcal{C}$ somewhat open injection and Y is a $\mathcal{D} - \mathcal{C}$ -space, then X is also a $\mathcal{D} - \mathcal{C}$ space.

Proof: let U be a any nonempty open set in X. Since f is $\theta - \mathcal{C}$ somewhat open there exists a nonempty open set V of Y such that $V_* \subseteq f(U_*)$. Again since Y is a $\mathcal{D}-\mathcal{C}$ -space, $V_* = Y$. So we have $Y \subseteq f(U_*)$. Now $X = f^{-1}(Y) \subseteq f^{-1}(f(U_*)) = U_*$ (since f is injective) and thus we have $U_* = X$. Consequently X is a $\mathcal{D} - \mathcal{C}$ space.

Theorem 3.4 Let $(X, \tau, \mathcal{C}_1)$ be a $\mathcal{D} - \mathcal{C}$ space and $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a surjective function. Then f is a $\theta - \mathcal{C}$ open iff it is $\theta - \mathcal{C}$ somewhat open function.

Proof: From the definition 2.9 of $\theta - \mathcal{C}$ open function and $\theta - \mathcal{C}$ somewhat open function it is clear that if a function is $\theta - \mathcal{C}$ open then it is $\theta - \mathcal{C}$ somewhat open. To prove the converse part, let $x \in X$ and U be any nbd. of x. Then $\exists\ O \in \tau$ such that $x \in O \subseteq U$.Since X is a $\mathcal{D} - \mathcal{C}$ space, $O_* = X$ and then we have $U_* = X$.If we consider Y as a nbd. of $f(x)$, then we get, $Y_* = Y = f(X) = f(U_*)$. Hence f is $\theta - \mathcal{C}$ open function.

Definition 3.5 Let $(X, \tau, \mathcal{C})$ be a convex topological space. A subset G of X is said to be a $\tau_* - \mathcal{C}$ closed if $(Int(G))_* = G$.

Definition 3.6 A convex topological space $(X, \tau, \mathcal{C})$ is said to be a $\theta - \mathcal{C}$ irreducible space if every pair of nonempty $\tau_* - \mathcal{C}$ closed subsets of X has a nonempty intersection.

Remark 3.7 Every $\mathcal{D} - \mathcal{C}$ space X is a $\theta - \mathcal{C}$ irreducible space.

Let H, G be two nonempty $\tau_* - \mathcal{C}$ closed sets.Then $H = (Int(H))_*$, $G = (Int(G))_*$. since $Int(H)$, $Int(G)$ are nonempty open sets and X is $\mathcal{D} - \mathcal{C}$ space, $(Int(H))_* = X$, $(Int(G))_* = X$. This shows that $H \bigcap G = X \neq \phi$. Hence X is $\theta - \mathcal{C}$ irreducible space.

The converse may not be true which follows from the next example.

Example 3.8 Consider the convex topological space $(X, \tau, \mathcal{C})$ where $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}.\{b, c\}\}$, $\mathcal{C} = \{\phi, X, \{a\}\}$.

Here $(Int\{a\})_* = \{a\}_* = x$, $(Int\{b\})_* = \phi_* = \phi$, $(Int\{c\})_* = \phi_* = \phi$,$(Int\{a, b\})_* = \{a\}_* = x$, $(Int\{b, c\})_* = \{b, c\}_* = \{b, c\}$,$(Int\{a, c\})_* = \{a\}_* = x$. So nonempty $\tau_* - \mathcal{C}$ closed sets are $\{b, c\}$,X and $\{b, c\} \bigcap X = \{b, c\} \neq \phi$. Thus $(X, \tau, \mathcal{C})$ is $\theta - \mathcal{C}$ irreducible space. Now $\{b, c\} \in \tau$ and $\{b, c\}_* = \{b, c\} \neq X$. Hence $(X, \tau, \mathcal{C})$ is not $\mathcal{D} - \mathcal{C}$ space .

Proposition 3.9 Let $(X, \tau, \mathcal{C})$ be a convex topological space which is $\tau - \mathcal{C}$ semi compatible. For any subset A of X , $Int(A))_*$ is $\tau_* - \mathcal{C}$ closed.

Proof: Let $V = (Int(A))_*$. We will show that $(Int(V))_* = V$.

Now $Int(V) \subseteq V$ which implies that $(Int(V))_* \subseteq V_* = ((Int(A)_*)_* = (Int(A))_*$ (since X is $\tau - \mathcal{C}$ semi compatible)$=V$.

Again let $x \in (Int(A))_*$ and $x \in U \in \tau$. Now $Int(A) \subseteq (Int(A))_*$ which implies that $Int(A) \subseteq Int((Int(A))_*$. $x \in (Int(A))_* \Rightarrow co(U) \cap Int(A) \neq \phi \Rightarrow co(U) \cap Int((Int(A))_*) \neq \phi$ $\Rightarrow x \in [Int((Int(A))_*)]_*$. Thus we have $V \subseteq (Int(V))_*$. Hence $V = (Int(V))_*$.

Theorem 3.10 Let $(X, \tau, \mathcal{C})$ be a convex topological space which is $\tau - \mathcal{C}$ semi compatible. Then $(X, \tau, \mathcal{C})$ is not a $\theta - \mathcal{C}$ irreducible space iff $\exists$ nonempty open subsets U and V of X such that $U_* \cap V_* = \phi$.

Proof: Let X be not a $\theta - \mathcal{C}$ irreducible space. Then there exists nonempty $\tau_* - \mathcal{C}$ closed sets A and B of X such that $A \cap B = \phi$. Since A and B are $\tau_* - \mathcal{C}$ closed sets, $(Int(A))_* = A$ and $(Int(B))_* = B$. Let $U = Int(A)$ and $V = Int(B)$. Then U and V are nonempty open sets such that $U_* \cap V_* = \phi$. Conversely let there exist a nonempty open sets U and V of X such that $U_* \cap V_* = \phi$. Let $A = U_*$ and $B = V_*$. Then $A = (Int(U))_*$ and $B = (Int(V))_*$ (by proposition3.9) are nonempty $\tau_* - \mathcal{C}$ closed sets of X such that $A \cap B = \phi$. Hence X is not $\theta - \mathcal{C}$ irreducible space.

Theorem 3.11 Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is $\theta - \mathcal{C}$ somewhat open injection where $(Y, \sigma, \mathcal{C}_2)$ is $\sigma - \mathcal{C}$ semi compatible. If Y is $\theta - \mathcal{C}$ irreducible space then X is also a $\theta - \mathcal{C}$ irreducible space.

Proof: Suppose that X is not a $\theta - \mathcal{C}$ irreducible space. Then there exists nonempty open sets U and V of X such that $U_* \cap V_* = \phi$. Since f is $\theta - \mathcal{C}$ somewhat open function there exists nonempty open sets G and H of Y such that $G_* \subseteq f(U_*)$ and $H_* \subseteq f(V_*)$. Since f is injective $G_* \cap H_* = \phi$. This shows that Y is not a $\theta - \mathcal{C}$ irreducible space-which is a contradiction. Hence X is a $\theta - \mathcal{C}$ irreducible space.

4 $\theta - \mathcal{C}$ somewhat continuous function

Definition 4.1 Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X, \tau, \mathcal{C}_1) \to Y, \sigma, \mathcal{C}_2)$ is said to be

1. $\theta-\mathcal{C}$ continuous function if for each $x \in X$ and each open nbd.V of $f(x)$, there exists an open nbd. U of x such that $f(U_*) \subseteq V_*$ and

2. $\theta-\mathcal{C}$ somewhat continuous function if $V \in \sigma$ and $f^{-1}(V) \neq \phi$, there exists nonempty open set U in X such that $U_* \subseteq f^{-1}(V_*)$.

Remark 4.2 From the above definition it follows that $\theta-\mathcal{C}$ continuity implies $\theta-\mathcal{C}$ somewhat continuity. But the converse is not always true which follows from the next example.

Example 4.3 Let $X = \{a,b,c\}$, $\tau = \{\phi, X, \{a,b\}, \{c\}\}$, $\mathcal{C}_1 = \{\phi, X, \{a,b\}, \{c\}\}$, $\sigma = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b.c\}\}, \mathcal{C}_2 = \{\phi, X, \{a\}, \{c\}\}$. Here the identity function $i : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ is $\theta-\mathcal{C}$ somewhat continuous function but not $\theta-\mathcal{C}$ continuous function. It is clear that i is $\theta-\mathcal{C}$ somewhat continuous function. Consider the point $b \in X$. Now $\{b,c\}$ is nbd. of $b = i(b)$ in $(X, \sigma, \mathcal{C}_2)$ and in $(X, \sigma, \mathcal{C}_2)$, $\{b,c\}_* = \{b,c\}$. Again nbd. of b in $(X, \tau, \mathcal{C}_1)$ are X and $\{a,b\}$. In this space $X_* = X$ and $\{a,b\}_* = \{a,b\}$. But $\{a,b\} \nsubseteq i^{-1}\{b,c\} = \{b,c\}$ and $X \nsubseteq i^{-1}\{b,c\} = \{b,c\}$. Consequently i is not $\theta-\mathcal{C}$ continuous function.

Theorem 4.4 Composition of two $\theta-\mathcal{C}$ somewhat continuous functions is again $\theta-\mathcal{C}$ somewhat continuous function.
Proof: Obvious.

Theorem 4.5 Let $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be a $\theta-\mathcal{C}$ somewhat continuous surjection. If $(X, \tau, \mathcal{C}_1)$ be a $\mathcal{D}-\mathcal{C}$ space then $(Y, \sigma, \mathcal{C}_2)$ is also a $\mathcal{D}-\mathcal{C}$ space.
Proof: Let $V(\neq \phi) \in \sigma$. Since f is surjective , $f^{-1}(V) \neq \phi$. As f is $\theta-\mathcal{C}$ somewhat continuous function, there exists $U(\neq \phi) \in \tau$ such that $U_* \subseteq f^{-1}(V_*)$. Now X is a $\mathcal{D}-\mathcal{C}$ space. So $U_* = X$. Hence $Y = f(X) = f(U_*) \subseteq ff^{-1}(V_*) = V_*$ i.e. we have $V_* = Y$. This shows that Y is a $\mathcal{D}-\mathcal{C}$ space.

Theorem 4.6 Let $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be a $\theta-\mathcal{C}$ somewhat continuous function where $(X, \tau, \mathcal{C}_1)$ is $\tau-\mathcal{C}$ semi compatible. If X is a $\theta-\mathcal{C}$ irreducible space then Y is also a $\theta-\mathcal{C}$ irreducible space.
Proof: Let Y be not $\theta-\mathcal{C}$ irreducible space. Then there exist nonempty open sets U and V in Y such that $U_* \cap V_* = \phi$. Since f is $\theta-\mathcal{C}$ somewhat continuous, there exist nonempty open sets G and H in X such that $G_* \subseteq f^{-1}(U_*)$ and $H_* \subseteq f^{-1}(V_*)$. This implies that $G_* \cap H_* \subseteq f^{-1}(U_*) \cap f^{-1}(V_*) = f^{-1}(U_* \cap V_*) = \phi$. This shows that X is not a $\theta-\mathcal{C}$ irreducible space-which is a contradiction. Hence Y is a $\theta-\mathcal{C}$ irreducible space.

Result 4.7 [5] Let $(X, \tau, \mathcal{C})$ be a convex topological space and $A \subseteq X$. Consider the convex topological space $(A, \tau_A, \mathcal{C}_A)$ where τ_A is subspace topology and $\mathcal{C}_A$ is relative convexity on A. Then for any subset B of A , $((B)_*)^{\tau_A} \subseteq B_*$.

Theorem 4.8 Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a $\theta-\mathcal{C}$ somewhat continuous function and A be a dense subset of X. Then $f : (A, \tau_A, \mathcal{C}_A) \to (Y, \sigma, \mathcal{C}_2)$ is also $\theta - \mathcal{C}$ somewhat continuous function.

Proof: Let $V \in \sigma$ such that $f^{-1}(V) \neq \phi$. Since $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is a $\theta-\mathcal{C}$ somewhat continuous function, there exists $U(\neq \phi) \in \tau$ such that $U_* \subseteq f^{-1}(V_*)$. A is dense in X implies that $A \cap U \neq \phi$. Now $U \cap A \in \tau_A$. By result 4.7 we have $(U \cap A)_*^{\tau_A} \subseteq (U \cap A)_* \subseteq U_*$. So $(U \cap A)_*^{\tau_A} \subseteq U_* \subseteq f^{-1}(V_*)$. This shows that $f : (A, \tau_A, \mathcal{C}_A) \to (Y, \sigma, \mathcal{C}_2)$ is also $\theta - \mathcal{C}$ somewhat continuous function.

Theorem 4.9 Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a function where $(Y, \sigma, \mathcal{C}_2)$ is $\mathcal{D} - \mathcal{C}$-space. Then f is $\theta - \mathcal{C}$ continuous function iff $\theta - \mathcal{C}$ somewhat continuous function.

Proof: For any function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ it is clear that $\theta - \mathcal{C}$ continuity $\Rightarrow \theta - \mathcal{C}$ somewhat continuity [by Remark 4.2].

Conversely let f be $\theta - \mathcal{C}$ somewhat continuous function and Y is a $\mathcal{D} - \mathcal{C}$-space.

Let $x \in X$ and V be open nbd. of $f(x)$ in $(Y, \sigma, \mathcal{C}_2)$. since Y is a $\mathcal{D} - \mathcal{C}$-space, $V_* = Y$. In $(X, \tau, \mathcal{C}_1)$, we take X as a open nbd. of x. Then clearly $f(X_*) = f(X) \subseteq Y = V_*$. Consequently f is $\theta - \mathcal{C}$ continuous function.

Result 4.10 Let $(X, \tau, \mathcal{C})$ be a convex topological space. Let $A \in \tau$. Then $(A_*)^{\tau} \subseteq (A_*)^{\tau_A}$.

Proof: Consider the convex topological space $(X, \tau_A, \mathcal{C}_A)$. We have to prove that $(A_*)^{\tau} \subseteq (A_*)^{\tau_A}$. Let $x \in A_*$ and let $V \in \tau_A$ such that $x \in V \in \tau_A$. Since $V \in \tau_A$, $V = A \cap U$ for some $U \in \tau$. This shows that $V \in \tau$. Now $x \in (A_*)^{\tau} \Rightarrow co(V) \cap A \neq \phi \Rightarrow co(V) \cap A \cap A \neq \phi \Rightarrow co_A(V) \cap A \neq \phi$ [by relative hull formula] $\Rightarrow x \in (A_*)^{\tau_A}$. Hence $(A_*)^{\tau} \subseteq (A_*)^{\tau_A}$.

Theorem 4.11 Let $(X, \tau, \mathcal{C})$ and $(Y, \sigma, \mathcal{C}_1)$ be two convex topological spaces. Let A be an open subset of X such that $f : (A, \tau_A, \mathcal{C}_A) \to (Y, \sigma, \mathcal{C}_1)$ is $\theta-\mathcal{C}$ somewhat continuous function and $f(A)$ is dense in Y. Then any extension F of f is $\theta - \mathcal{C}$ somewhat continuous function.

Proof: Let U be any open set in Y such that $F^{-1}(U) \neq \phi$. Since $f(A)$ is dense in Y, $U \cap f(A) \neq \phi$ and $F^{-1}(U) \cap A \neq \phi$. That is $f^{-1}(U) \cap A \neq \phi$. Since $f : (A, \tau_A, \mathcal{C}_A) \to (Y, \sigma, \mathcal{C}_1)$ is $\theta - \mathcal{C}$ somewhat continuous function and $U \in \sigma$ with $f^{-1} \neq \phi$, there exists

$V \in \tau_A$ with $V \neq \phi$ such that $(V_*)^{\tau_A} \subseteq f^{-1}(U_*)$——-(1). It is clear that V is open in X as $A \in \tau$. Thus by result 4.10 we have $(V_*)^{\tau} \subseteq (V_*)^{\tau_A}$. Consequently from (1) we have $(V_*)^{\tau} \subseteq (V_*)^{\tau_A} \subseteq f^{-1}(U_*) \subseteq F^{-1}(U_*)$. This shows that F is $\theta - \mathcal{C}$ somewhat continuous.

Theorem 4.12 Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. Suppose $X = A \cup B$ where A and$B$ are open subsets of X and $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is a function such that $f\mid_A$ and $f\mid_B$ are $\theta - \mathcal{C}$ somewhat continuous function. Then f is $\theta - \mathcal{C}$ somewhat continuous function.

Proof: Let $U \in \sigma$ such that $f^{-1}(U) \neq \phi$. Then $(f\mid_A)^{-1}(U) \neq \phi$ or $(f\mid_B)^{-1}(U) \neq \phi$ or both $(f\mid_A)^{-1}(U) \neq \phi$ and $(f\mid_B)^{-1}(U) \neq \phi$.

Case 1. $(f\mid_A)^{-1}(U) \neq \phi$. Since $f\mid_A$: $(A, \tau_A, \mathcal{C}_A) \to (Y, \sigma, \mathcal{C}_2)$ is $\theta - \mathcal{C}$ somewhat continuous, there exists $V \in \tau_A$ with $V \neq \phi$ such that $(V_*)^{\tau_A} \subseteq (f\mid_A)^{-1}(U_*) \subseteq f^{-1}(U_*)$. As $V \in \tau$ by result 4.10 $(V_*)^{\tau} \subseteq (V_*)^{\tau_A} \subseteq f^{-1}(U_*)$. This implies that $f : (X, \tau, \mathcal{C}) \to (Y, \sigma, \mathcal{C}_2)$ is $\theta - \mathcal{C}$ somewhat continuous function. Similarly for the other case.

Definition 4.13 Let $(X, \mathcal{C})$ be a convex structure and let τ_1 and τ_2 be two topology on X. Then τ_1 is said to be $\theta - \mathcal{C}$ weakly equivalent to τ_2 provided if $U \in \tau_1$ and $U \neq \phi$, then there exists a nonempty set $V \in \tau_2$ such that $(V_*)^{\tau_2} \subseteq (U_*)^{\tau_1}$ and if $P \in \tau_2$ and $P \neq \phi$, then there exists a nonempty set $Q \in \tau_1$ such that $(Q_*)^{\tau_1} \subseteq (P_*)^{\tau_2}$.

Note 4.14 Consider the identity function $i : (X, \tau_1, \mathcal{C}) \to (X, \tau_2, \mathcal{C})$ and let τ_1 and τ_2 be weakly equivalent. Let $V \in \tau_2$ such that $i^{-1}(V) = V \neq \phi$. Then there exists $U \in \tau_1$ with $U \neq \phi$ such that $(U_*)^{\tau_1} \subseteq (V_*)^{\tau_2} \Rightarrow (U_*)^{\tau_1} \subseteq (V_*)^{\tau_2} = i^{-1}((V_*)^{\tau_2})$. This shows that i is $\theta - \mathcal{C}$ somewhat continuous function. Similarly we can show that $i : (X, \tau_2, \mathcal{C}) \to (X, \tau_1, \mathcal{C})$ is also $\theta - \mathcal{C}$ somewhat continuous function.

Conversely if $i : (X, \tau_1, \mathcal{C}) \to (X, \tau_2, \mathcal{C})$ is $\theta - \mathcal{C}$ somewhat continuous function in both directions then τ_1 and τ_2 are $\theta - \mathcal{C}$ weakly equivalent.

Theorem 4.15 Let $f : (X, \tau_1, \mathcal{C}) \to (Y, \sigma, \mathcal{C})$ be $\theta - \mathcal{C}$ somewhat continuous function. Let τ_2 be a topology on X which is $\theta - \mathcal{C}$ weakly equivalent to τ_1. Then $f : (X, \tau_2, \mathcal{C}) \to (y, \sigma, \mathcal{C})$ is $\theta - \mathcal{C}$ somewhat continuous function.

Proof: Obvious.

Theorem 4.16 Let $f : (X, \tau, \mathcal{C}) \to (y, \sigma, \mathcal{C})$ be $\theta - \mathcal{C}$ somewhat continuous function. Let σ_1 be a topology on Y which is $\theta - \mathcal{C}$ weakly equivalent to σ. Then $f : (X, \tau, \mathcal{C}) \to (y, \sigma_1, \mathcal{C})$ is also $\theta-\mathcal{C}$ continuous function.

Proof: Obvious.

5 Comparison between θ–$\mathcal{C}$ somewhat continuous function and somewhat continuous function

In this article we will show that a $\theta-\mathcal{C}$ somewhat continuous function is not necessarily a somewhat continuous function and vice versa. Here we also show that there exists a special type of convex topological space in which the above two concept coincides.

Example 5.1 Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{b\},\{d\},\{b,d\}\}$, $\tau_1=\{\phi,X,\{a,b\},\{c\},\{a,b,c\}\}$, $\mathcal{C}=\mathcal{C}_1=\{\phi,X\}$ and consider the function $f:(X,\tau,\mathcal{C})\to(X,\tau_1,\mathcal{C}_1)$ defined by $f(a)=c$, $f(b)=a$ $f(c)=c$ $f(d)=b$. Here f is $\theta-\mathcal{C}$ some what continuous function but not somewhat continuous function. Note that for any $U\in\tau_1$, in the convex topological space $(X,\tau_1,\mathcal{C}_1)$, $U_*=X$. Now $\{c\}\in\tau_1$ and $f^{-1}(\{c\})=\{a,c\}\neq\phi$, but there is no $V\in\tau$ such that $V\subseteq f^{-1}(\{c\})$.

Example 5.2 Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\},\{b\},\{a,b\}\}$, $\mathcal{C}=\{\phi,X\}$, $\mathcal{C}_1=\{\phi,X,\{a\}\}$ and consider the identity function $f:(X,\tau,\mathcal{C})\to(X,\tau_1,\mathcal{C}_1)$. It is clear that f is somewhat continuous function. Now $\{b\}\in\tau_1$ and in the convex topological space $(X,\tau_1,\mathcal{C}_1)$, $\{b\}_*=\{a,c\}$. Again in the convex topological space $(X,\tau,\mathcal{C})$ for any $U\in\tau$, $U_*=X$. Thus for $\{b\}\in\tau_1$, there is no $V\in\tau$ such that $V_*\subseteq f^{-1}(\{b\}_*)$. Consequently f is not $\theta-\mathcal{C}$ somewhat continuous function.

Definition 5.3 A convex topological space $(X,\tau,\mathcal{C})$ is said to be strongly $\mathcal{C}$-regular space if for any nonempty set $U\in\tau$ there exists a nonempty set $V\in\tau$ such that $V_*\subseteq U$.

Example 5.4 Let us consider the convex topological space $(X,\tau,\mathcal{C})$, where (X,τ) is discrete topological space and $\mathcal{C}$ is defined by $\mathcal{C}=\{\phi,X\}\cup\{\{x\}:x\in X\}$. then for any $U\in\tau$, $U_*=U$ and thus $(X,\tau,\mathcal{C})$,is strongly $\mathcal{C}$-regular space.

Example 5.5 Let R denote the set of reals and τ_U be the usual topology on R. Here the convexity $\mathcal{C}$ is defined as follows : A set $C\in\mathcal{C}$ iff for any two points $a,b\in C$, the convex combination of a, b must be in C i.e. $\mathcal{C}$ is the standard convexity on R. Then it is clear that $(R,\tau_U,\mathcal{C})$ is a strongly $\mathcal{C}$-regular space.

Example 5.6 Any locally convex space (X, τ) is strongly $\mathcal{C}$-regular space. Already we have proved in [5] that in a locally convex space, if $A = \overline{A}$ then $A = \overline{A} = A_*$. Let V be any nonempty open set in X. Since (X, τ) is regular space, there exists $W \in \tau$ with $W \neq \phi$, such that $\overline{W} \subseteq V$. So we have $(\overline{W})_* = \overline{W} \subseteq V$. This shows that $W_* \subseteq (\overline{W})_* \subseteq V$. So we see that for any nonempty open set V , there exists a nonempty open set W such that $W_* \subseteq V$. Consequently (X, τ) is strongly $\mathcal{C}$-regular space.

Theorem 5.7 Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a $\theta - \mathcal{C}$ somewhat continuous function which is onto. If $(Y, \sigma, \mathcal{C}_2)$ be strongly $\mathcal{C}$-regular space, then f is a somewhat continuous function.
Proof: Let $U \in \sigma$ with $U \neq \phi$. Then $f^{-1}(U) \neq \phi$. Now $U \in \tau$ and $(Y, \sigma, \mathcal{C}_2)$ is strongly $\mathcal{C}$-regular space so there exists $W \in \sigma$ with $W \neq \phi$ such that $W_* \subseteq U$. Since f is $\theta - \mathcal{C}$ somewhat continuous function and $f^{-1}(W) \neq \phi$, there exists $V \in \tau$ with $V \neq \phi$ such that $V_* \subseteq f^{-1}(W_*)$. This implies that $V \subseteq V_* \subseteq f^{-1}(W_*) \subseteq f^{-1}(U)$. Consequently f is somewhat continuous function.

Theorem 5.8 $f : (X, \tau, \mathcal{C}) \to (Y, \sigma, \mathcal{C}_1)$ be a somewhat continuous function. If $(X, \tau, \mathcal{C})$ be a strongly $\mathcal{C}$-regular space, then f is $\theta - \mathcal{C}$ somewhat continuous function.
Proof: Let $U \in \sigma$ and $f^{-1}(U) \neq \phi$. Since f is somewhat continuous function there exists $V \in \tau$ with $V \neq \phi$ such that $V \subseteq f^{-1}(U)$. Again $(X, \tau, \mathcal{C})$ is a strongly $\mathcal{C}$-regular space so for $V \in \tau$ there exists $W \in \tau$ with $W \neq \phi$ such that $W_* \subseteq V$. This shows that $W_* \subseteq V \subseteq f^{-1}(U) \subseteq f^{-1}(U_*)$. Therefore f is $\theta - \mathcal{C}$ somewhat continuous function.

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